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Autor:	Ballico, Edoardo
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# ON COUSIN-I COMPLEX SPACES

by Edoardo BALLICO<sup>1</sup>)

## SUMMARY

We study the properties of a complex space such that every closed analytic subspace has the 1-Cousin property. Under mild hypotheses we prove that X is a Stein space.

G. Berg in [2] proved very easily the following result: Let X be an open subset of a two-dimensional Stein space Y. If in X every 1-Cousin problem is solvable, then X is a domain of holomorphy in Y. It is not difficult to modify his proof and obtain that, under the stated assumption, X is always a Stein space.

In this note we want to consider this problem also for higher dimensional complex spaces. The proofs are always by induction on the dimension; they are similar to the proofs in the two dimensional case. This is also a partial generalization of my previous [1], theorem 2. We say that a complex space X is Cousin-I or has the 1-Cousin property if every 1-Cousin problem on X is solvable. Any Stein space is a Cousin-I space. We denote by O(X) the algebra of holomorphic functions on a complex space X. For each  $f \in O(X)$  we put

$$\{x \in X : f(x) = 0\} = : V(f),$$

an analytic subset of X. In this paper we consider for simplicity only complex spaces with bounded Zariski tangent dimension.

LEMMA 1. Let X be a complex space with the 1-Cousin property and  $f \in O(X)$  such that for each  $x \in X$  the germ of f around x is a non-zero divisor in  $\mathcal{O}_{X,x}$ . We put

with

$$Z := (V(f), \mathcal{O}_Z)$$

$$\mathcal{O}_Z := \mathcal{O}_X / f \mathcal{O}_{X|V(f)}.$$

Then the natural map  $p: O(X) \rightarrow O(Z)$  is surjective.

<sup>&</sup>lt;sup>1</sup>) The author is member of G.N.S.A.G.A. of C.N.R. (Italy).

*Proof*: Let  $\{U_i\}$  be a family of Stein open subsets of X which covers V(f) and such that for each index i we have  $U_i \cap V(f) \neq \emptyset$ .

We consider the exact sequence of sheaves on X

$$0 \to \mathcal{O}_X \xrightarrow{\iota} \mathcal{O}_X \to \mathcal{O}_Z \to 0 \tag{1}$$

where t is the multiplication by f. It is exact because f is a non zero-divisor in  $\mathcal{O}_{X,x}$  for each  $x \in X$ . We take  $h \in O(Z)$ . From theorem B of Cartan-Serre it follows that for every i the restriction of h to  $V(f) \cap U_i$  can be extended to a holomorphic function  $f \in O(U_i)$ . If we have  $U_i \cap U_j \neq \emptyset$ , from the restriction of the exact sequence (1) to  $U_i \cap U_j$ , we obtain that  $f_i - f_j = g_{ij} f$ , where  $g_{ij} \in O(U_i \cap U_j)$  is uniquely determined because f is not a zero-divisor of  $O(U_i \cap U_j)$ .

We take the covering of X given by  $X \setminus V(f)$  and  $\{U_i\}$ . As data for the 1-Cousin problem in X we take the costant 1 on  $X \setminus V(f)$  and the meromorphic function  $f_i/f$  on X. Again  $f_i/f$  is a meromorphic function on  $U_i$ , because f is not a zero-divisor on  $\mathcal{O}_{X,x}$  for each  $x \in U_i$ .

We obtain a meromorphic function g on X which satisfies the thesis of the 1-Cousin problem. The function gf is holomorphic on X and gives an extension of h to X.

G. Berg in [2] lemma, gave a proof more or less of the lemma above, if V(f) is a Stein space. He uses a strong theorem of Siu about the existence of a Stein open neighborhood of any Stein analytic subspace in any complex space. Our proof is similar, but does not use this strong theorem and is therefore more elementary.

THEOREM 1. Let X be a complex n-dimensional space, such that O(X) gives local coordinates (in particular it is holomorphically spreadable). Suppose that X and every closed analytic subspace have the 1-Cousin property. Suppose that X is either reduced or Cohen-Macaulay. Then X is a Stein space.

*Proof:* The proof is by induction on  $n = \dim X$ . If we have  $n \le 1$ , then X is a Stein space because it does not contain any positive dimensional analytic compact subspace. Suppose  $n \ge 2$ .

We put as in [6]  $S_d = \{x \in X : \operatorname{prof}_x (\mathcal{O}_{X,x}) \leq d\}$ . By [6] theorem 1.11,  $S_d$  is an analytic closed subspace of X with dim  $S_d \leq d$ . For each  $f \in O(X)$ , we have dim  $(V(f) \cap S_{k+1}) \leq k$  for each integer k if and only if  $f_x$  is a non zero-divisor in  $\mathcal{O}_{X,x}$  for each  $x \in X$  by [6], Corollary 1.18.

We may assume that X has no isolated point. Therefore, both under the hypotheses that X is reduced or that X is Cohen-Macauley, we obtain  $S_o = \emptyset$ .

By a simple application of Baire's cathegory theorem, we obtain  $f \in O(X)$  such that h has two different values on each irreducible component of dimension k of  $S_k$ .

First we want to prove that X is holomorphically separated. We fix two distinct points  $x, y \in X$ . If we have  $f(x) \neq f(y)$ , the result is trivially true. Suppose f(x) = f(y). By considering instead of f the function f - f(x), we may suppose that f vanishes at x. Therefore we suppose x,  $y \in V(f)$  and we put  $Z := (V(f), \mathcal{O}_Z)$  with

$$\mathcal{O}_Z := \mathcal{O}_X / f \mathcal{O}_{X|V(f)}.$$

a) If X is reduced, V(f) with the reduced structure is by the inductive hypothesis a Stein space. But a complex space Y is a Stein space if and only if  $Y_{red}$  is a Stein space. Therefore Z is a Stein space, too.

b) If X is Cohen-Macaulay, then also Z is Cohen-Macauley. In fact for each  $x \in X$  with f(x) = 0, we have

$$\operatorname{prof}\left(\mathcal{O}_{X,x}/f \mathcal{O}_{X,x}\right) = \operatorname{prof}\left(\mathcal{O}_{X,x}\right) - 1$$

by [6], lemma 1.2, and

$$\dim \left(\mathcal{O}_{X,x}/f \mathcal{O}_{X,x}\right) = \dim \mathcal{O}_{X,x} - 1.$$

From the inductive hypothesis it follows that Z is a Stein space.

Therefore under both assumptions we have proved that Z is a Stein space. In particular there exists a  $g \in O(Z)$  such that  $g(x) \neq g(y)$ . From lemma 1 it follows that there exists  $G \in O(X)$  which extends g. In particular we have  $G(x) \neq G(y)$ , proving that X is holomorphically separated.

Let  $\chi$  be a character of O(X);  $\chi$  is a multiplicative functional from O(X) to C. By [5], Corollary pag. 222, or [4], pag. 182, to prove that X is a Stein space it is sufficient to demonstrate that  $\chi$  is a valuation in a point of X. We take  $h \in O(X)$ as before, i.e. such that h has two different values on each irreducible component of dimension k of  $S_k$ . We put  $f = h - \chi(h)$ . We put H: = ker ( $\chi$ ). H is a maximal ideal of O(X). The function f is in H. We define Z as above. Z is again a Stein space by the inductive hypothesis. From lemma 1 and the exact sequence (1) in lemma 1, we obtain an exact sequence

$$0 \to f \ O(X) \to O(X) \to O(Z) \to 0$$
(2)

Since  $f \in \text{ker}(\chi)$ ,  $\chi$  induces a character  $\chi'$  on O(Z). Since Z is a Stein space,  $\chi'$  is induced by the valuation in a point  $x \in Z \subset X$ . If  $\{g_i\} \in O(Z)$  generate ker  $(\chi')$ 

and  $p(f_i) = g_i$ , the functions  $f_i$  and f generate  $H = \text{ker}(\chi)$ . Therefore, for every  $m \in H$  we have m(x) = 0. Since M is a maximal ideal,  $\chi$  is the valuation at the point x.

If dim X = 2, then it is sufficient to assume the 1-Cousin property for X. In fact its 1-dimensional subspace Z in the proof above is a Stein space because it has no compact, analytic subspace of positive dimension. If X is Cohen-Macaulay, it follows from lemma 1 that it is sufficient to assume that X and every closed analytic subspace Z of X given by global equations  $f_1, ..., f_k$  and with

$$\mathcal{O}_{\mathbf{Z}} := \mathcal{O}_{\mathbf{X}} / (f_1, ..., f_k) \mathcal{O}_{\mathbf{X} | \mathbf{Z}}$$

are Cousin-I space.

If in the theorem above we omit the condition that X has local coordinates given by global functions, we obtain only that every character is given by a valuation at a point of X. An easy inductive argument shows also that for every character  $\chi$  of O(X), ker ( $\chi$ ) is finitely generated. Therefore we obtain the following

**PROPOSITION 1.** Let X be a holomorphically spreadable complex space such that every closed analytic subspace has the 1-Cousin property. Suppose that X is reduced or Cohen-Macaulay. Then

- a) X is holomorphically separated,
- b) every character  $\chi$  of O(X) is a valuation at a point of X and ker  $(\chi)$  is finitely generated.

THEOREM 2. Let X be a complex reduced space which is a relatively compact open subset of a holomorphically separated, reduced complex space Y. Then X is a Stein space if and only if any closed analytic subspace, with its reduced structure, has the 1-Cousin property.

*Proof*: The "only if" part is well-known. We have dim  $X = n < +\infty$  and we use induction on n. For  $n \leq 1$  the result is well-known. Now suppose  $n \geq 2$ . Let  $\{x_n\}$  be a sequence in X without accumulation points in X. We may suppose, eventually extracting a subsequence, that  $\{x_n\}$  converges to a point  $y \in \mathcal{O}(X)$ . If we take into account the proof of [3], theorem 1.3, it remains only to prove that there exist

$$f_1, ..., f_k \in O(Y)$$

with only y as a common zero and such that there exist

 $g_1, ..., g_k \in O(X)$ 

with  $f_1g_1 + ... + f_kg_k = 1$  on X. In fact one of the  $g_i$ 's must be unbounded on the sequence  $\{x_n\}$ . As in the proof of theorem 1 we take  $f \in O(Y)$  a non zerodivisor in  $\mathcal{O}_{Y,x}$  for each  $x \in Y$  and vanishing in the point y. By the inductive hypothesis  $V(f) \cap X$  with its reduced structure is a Stein space. We define W:  $= (V(f), \mathcal{O}_W)$  with

$$\mathcal{O}_{W}$$
: =  $\mathcal{O}_{Y}/f \mathcal{O}_{Y|V(f)}$ 

and  $Z := W_{|X \wedge W}$  is a Stein space, too. Let  $p : O(X) \rightarrow O(Z)$  be the projection. From lemma 1 it follows that p is surjective. Let  $h_1, ..., h_k$  be holomorphic functions on W such that y is their unique common zero. Since Z is a Stein space and  $h_1, ..., h_k$  have no common zero in Z, there exist

$$g_1, ..., g_k \in O(Z)$$

such that we have  $g_1h_{1|Z} + ... + g_kh_{k|Z} = 1$  on Z. We take  $G_i \in O(X)$  such that  $p(G_i)$  is  $g_i$ . By the exact sequence (2) we obtain that there exists  $G \in O(X)$  such that we have  $Gf + ... + G_kh_k = 1$  on X. The proof is now finished.

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Edoardo Ballico

Scuola Normale Superiore Pisa Italia

