Proof: Let $\{U_i\}$ be a family of Stein open subsets of X which covers $V(f)$ and such that for each index i we have $U_i \cap V(f) \neq \emptyset$.

We consider the exact sequence of sheaves on X

$$0 \to \mathcal{O}_X \xrightarrow{t} \mathcal{O}_X \to \mathcal{O}_Z \to 0 \tag{1}$$

where t is the multiplication by f. It is exact because f is a non zero-divisor in $\mathcal{O}_{X,x}$ for each $x \in X$. We take $h \in O(Z)$. From theorem B of Cartan-Serre it follows that for every i the restriction of h to $V(f) \cap U_i$ can be extended to a holomorphic function $f \in O(U_i)$. If we have $U_i \cap U_j \neq \emptyset$, from the restriction of the exact sequence (1) to $U_i \cap U_j$, we obtain that $f_i - f_j = g_{ij} f$, where $g_{ij} \in O(U_i \cap U_j)$ is uniquely determined because f is not a zero-divisor of $O(U_i \cap U_j)$.

We take the covering of X given by $X \backslash V(f)$ and $\{U_i\}$. As data for the 1-Cousin problem in X we take the costant 1 on $X \backslash V(f)$ and the meromorphic function f_i/f on X. Again f_i/f is a meromorphic function on U_i, because f is not a zero-divisor on $\mathcal{O}_{X,x}$ for each $x \in U_i$.

We obtain a meromorphic function g on X which satisfies the thesis of the 1-Cousin problem. The function gf is holomorphic on X and gives an extension of h to X. □

G. Berg in [2] lemma, gave a proof more or less of the lemma above, if $V(f)$ is a Stein space. He uses a strong theorem of Siu about the existence of a Stein open neighborhood of any Stein analytic subspace in any complex space. Our proof is similar, but does not use this strong theorem and is therefore more elementary.

THEOREM 1. *Let X be a complex n-dimensional space, such that $O(X)$ gives local coordinates (in particular it is holomorphically spreadable). Suppose that X and every closed analytic subspace have the 1-Cousin property. Suppose that X is either reduced or Cohen-Macaulay. Then X is a Stein space.*

Proof: The proof is by induction on $n = \dim X$. If we have $n \leqslant 1$, then X is a Stein space because it does not contain any positive dimensional analytic compact subspace. Suppose $n \geqslant 2$.

We put as in [6] $S_d = \{x \in X : \mathrm{prof}_x(\mathcal{O}_{X,x}) \leqslant d\}$. By [6] theorem 1.11, S_d is an analytic closed subspace of X with $\dim S_d \leqslant d$. For each $f \in O(X)$, we have $\dim(V(f) \cap S_{k+1}) \leqslant k$ for each integer k if and only if f_x is a non zero-divisor in $\mathcal{O}_{X,x}$ for each $x \in X$ by [6], Corollary 1.18.

We may assume that X has no isolated point. Therefore, both under the hypotheses that X is reduced or that X is Cohen-Macauley, we obtain $S_o = \emptyset$.

By a simple application of Baire's cathegory theorem, we obtain $f \in O(X)$ such that h has two different values on each irreducible component of dimension k of S_k.

First we want to prove that X is holomorphically separated. We fix two distinct points $x, y \in X$. If we have $f(x) \neq f(y)$, the result is trivially true. Suppose $f(x) = f(y)$. By considering instead of f the function $f - f(x)$, we may suppose that f vanishes at x. Therefore we suppose $x, y \in V(f)$ and we put $Z := (V(f), \mathcal{O}_Z)$ with

$$\mathcal{O}_Z := \mathcal{O}_X / f\, \mathcal{O}_{X|V(f)} \cdot$$

a) If X is reduced, $V(f)$ with the reduced structure is by the inductive hypothesis a Stein space. But a complex space Y is a Stein space if and only if Y_{red} is a Stein space. Therefore Z is a Stein space, too.

b) If X is Cohen-Macaulay, then also Z is Cohen-Macauley. In fact for each $x \in X$ with $f(x) = 0$, we have

$$\operatorname{prof}(\mathcal{O}_{X,x}/f\, \mathcal{O}_{X,x}) = \operatorname{prof}(\mathcal{O}_{X,x}) - 1$$

by [6], lemma 1.2, and

$$\dim(\mathcal{O}_{X,x}/f\, \mathcal{O}_{X,x}) = \dim \mathcal{O}_{X,x} - 1 .$$

From the inductive hypothesis it follows that Z is a Stein space.

Therefore under both assumptions we have proved that Z is a Stein space. In particular there exists a $g \in O(Z)$ such that $g(x) \neq g(y)$. From lemma 1 it follows that there exists $G \in O(X)$ which extends g. In particular we have $G(x) \neq G(y)$, proving that X is holomorphically separated.

Let χ be a character of $O(X)$; χ is a multiplicative functional from $O(X)$ to $\mathbf{C}$. By [5], Corollary pag. 222, or [4], pag. 182, to prove that X is a Stein space it is sufficient to demonstrate that χ is a valuation in a point of X. We take $h \in O(X)$ as before, i.e. such that h has two different values on each irreducible component of dimension k of S_k. We put $f = h - \chi(h)$. We put $H := \ker(\chi)$. H is a maximal ideal of $O(X)$. The function f is in H. We define Z as above. Z is again a Stein space by the inductive hypothesis. From lemma 1 and the exact sequence (1) in lemma 1, we obtain an exact sequence

$$0 \to f\, O(X) \to O(X) \to O(Z) \to 0 \tag{2}$$

Since $f \in \ker(\chi)$, χ induces a character χ' on $O(Z)$. Since Z is a Stein space, χ' is induced by the valuation in a point $x \in Z \subset X$. If $\{g_i\} \in O(Z)$ generate $\ker(\chi')$

and $p(f_i) = g_i$, the functions f_i and f generate $H = \ker(\chi)$. Therefore, for every $m \in H$ we have $m(x) = 0$. Since M is a maximal ideal, χ is the valuation at the point x. □

If $\dim X = 2$, then it is sufficient to assume the 1-Cousin property for X. In fact its 1-dimensional subspace Z in the proof above is a Stein space because it has no compact, analytic subspace of positive dimension. If X is Cohen-Macaulay, it follows from lemma 1 that it is sufficient to assume that X and every closed analytic subspace Z of X given by global equations $f_1, ..., f_k$ and with

$$\mathcal{O}_Z := \mathcal{O}_X/(f_1, ..., f_k)\, \mathcal{O}_{X|Z}$$

are Cousin-I space.

If in the theorem above we omit the condition that X has local coordinates given by global functions, we obtain only that every character is given by a valuation at a point of X. An easy inductive argument shows also that for every character χ of $O(X)$, $\ker(\chi)$ is finitely generated. Therefore we obtain the following

PROPOSITION 1. *Let X be a holomorphically spreadable complex space such that every closed analytic subspace has the 1-Cousin property. Suppose that X is reduced or Cohen-Macaulay. Then*

a) *X is holomorphically separated,*

b) *every character χ of $O(X)$ is a valuation at a point of X and* $\ker(\chi)$ *is finitely generated.*

THEOREM 2. *Let X be a complex reduced space which is a relatively compact open subset of a holomorphically separated, reduced complex space Y. Then X is a Stein space if and only if any closed analytic subspace, with its reduced structure, has the 1-Cousin property.*

Proof: The "only if" part is well-known. We have $\dim X = n < +\infty$ and we use induction on n. For $n \leqslant 1$ the result is well-known. Now suppose $n \geqslant 2$. Let $\{x_n\}$ be a sequence in X without accumulation points in X. We may suppose, eventually extracting a subsequence, that $\{x_n\}$ converges to a point $y \in \mathcal{O}(X)$. If we take into account the proof of [3], theorem 1.3, it remains only to prove that there exist

$$f_1, ..., f_k \in O(Y)$$

with only y as a common zero and such that there exist

$$g_1, ..., g_k \in O(X)$$