2. Pólya's proof of Descartes' theorem

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Gauss-Bonnet formula for smooth manifolds; an excellent account of the development in this direction is to be found in the article by Chern ([13]; see especially formula (4) on p. 343).

2. PÓLYA'S PROOF OF DESCARTES' THEOREM

We start from the position that Euler's formula for a polyhedral 2-sphere S^2 is known; that is to say, if P is a polyhedron homeomorphic to S^2 with V vertices, E edges and F faces, then

$$V - E + F = 2. \tag{2.1}$$

In Figure 1 (a), for example, $V = 4$, $E = 6$, $F = 4$. Thus $4 - 6 + 4 = 2$, verifying (2.1). Euler's formula is discussed in many elementary books on polyhedra and many proofs have been given. The book by Courant and Robbins, *What is Mathematics?* [4] contains a proof using networks. Pólya's book, *Mathematics and Plausible Reasoning*, Vol. I, [1], has a sequence of problems that leads the reader to a proof. Lakatos' *Proofs and Refutations* [8] is cleverly written in the format of a dialogue between a mathematics teacher and his extremely bright students (who continually find counterexamples to the proposed theorems). The "general" proof must be attributed to Poincaré [10] who, as explained in the Introduction, proved that the generalized Euler-Poincaré characteristic is a topological invariant which takes the value 2 on any even-dimensional sphere.

We now show how Pólya deduced Descartes theorem from (2.1); this argument is essentially that given in [2].

Let P be a polyhedron homeomorphic to S^2, subdivided into vertices, edges and faces in such a way that every edge is incident with exactly two faces. Number the vertices $1, 2, \ldots, V$ and let the sum of the plane face angles at the i-th vertex be σ_i. Then the angular defect at the i-th vertex is

$$\delta_i = 2\pi - \sigma_i .$$

Note that δ_i will be positive if P is convex, but that, in general, δ_i may be negative or zero. Let

$$\Delta = \sum_{i=1}^{V} \delta_i .$$

We want to show that $\Delta = 4\pi$.

Proceed by numbering the faces 1, 2, ..., F and let S_j be the number of sides [1]) of the j-th face. Then

$$(S_1-2)\,\pi + (S_2-2)\,\pi + (S_3-2)\,\pi + \ldots + (S_F-2)\,\pi$$
$$= V\,(2\pi) - \sum_{i=1}^{V} \delta_i = V\,(2\pi) - \Delta\,.$$

Rearranging the terms on the left yields

$$\pi\left(\sum_{j=1}^{F} S_j\right) - 2\pi F = 2\pi V - \Delta\,. \tag{2.2}$$

Now, since the total number of sides of the faces which make up the polyhedron P is twice the number of edges, E, on P, we have $\sum_{j=1}^{F} S_j = 2E$, so that

$$\pi\,(2E) - 2\pi F = 2\pi V - \Delta$$

or

$$\Delta = 2\pi V - 2\pi E + 2\pi F$$
$$= 2\pi\,(V-E+F)\,.$$

But, by Euler's formula (2.1), $V - E + F = 2$. Thus

$$\Delta = 2\pi\,(2) = 4\pi\,.$$

Our first observation is that Pólya's argument immediately generalizes to arbitrary 2-dimensional polyhedra (in the topologists' sense!). Thus let P be any 2-dimensional polyhedron, subdivided into vertices, edges and faces in such a way that every edge is incident with exactly two faces. Define the *Euler characteristic*, $\chi\,(P)$, by

$$\chi\,(P) = V - E + F\,, \tag{2.3}$$

where P has V vertices, E edges and F faces. Define the total angular defect Δ as above; that is

$$\Delta = \sum_{i=1}^{V} \delta_i\,,$$

[1]) It is very important to the understanding of this proof to distinguish between the meaning of a *side* and an *edge*. If a line segment joining two vertices is considered in relation to a face, to whose boundary it belongs, it is called a *side* of that face; if it is considered in relation to the whole polyhedron (forming the common boundary of two neighboring faces) it is called an *edge* of that polyhedron. Thus we see that we may think of the polyhedron as being formed by taking the individual faces and joining the sides of the faces to each other in pairs so that each pair then becomes a single edge of the polyhedron.

where δ_i is the sum of the plane face angles at the i-th vertex and $\delta_i = 2\pi - \sigma_i$. Then Pólya's argument immediately yields the theorem

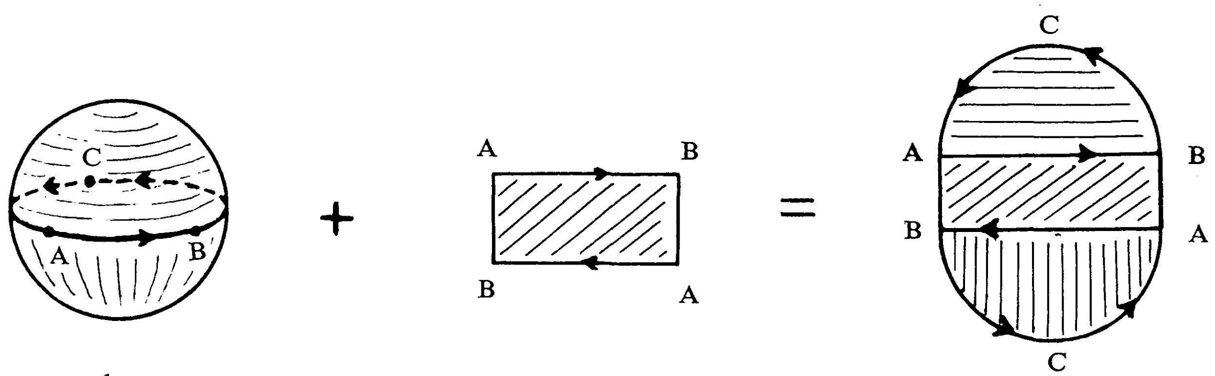
THEOREM 1. $\Delta(P) = 2\pi\chi(P)$.

A polyhedron P of the type discussed in this theorem is described in the literature of topology as a two-dimensional *pseudomanifold*. Included in this category is the family of *closed surfaces*. If S is such a surface we may take a *rectilinear model* of S, that is, a polyhedron P, homeomorphic to S, and subdivided into vertices, edges and faces as above. Closed surfaces are either *orientable* or non-orientable. An orientable closed surface of *genus* g $(g \geqslant 0)$ may be thought of as formed by attaching g handles to a sphere S^2. Thus if $g = 0$ we have the sphere; if $g = 1$ we have the torus; if $g = 2$ we have the double torus... In general, for an orientable surface S of genus g,

$$\chi(S) = 2 - 2g. \tag{2.4}$$

Observe that $\chi = 2$ for all of the models displayed in Figure 1. When the manifold is homeomorphic with a torus $\chi = 0$. Figure 2 (a) serves to illustrate this example of Theorem 1. Notice that the figure has 14 vertices, 29 edges and 15 faces (2 triangles and 13 quadrilaterals). The computation for the sum of the angular deficiencies produced at all of the 14 vertices may be verified to be $2\pi\chi$. This computation may be displayed, instructively, as follows:

$$\begin{aligned}
\Delta &= 14(2\pi) - \{2(3-2)\pi + 13(4-2)\pi\} \\
&= \underset{V}{14(2\pi)} - \underset{E}{29(2\pi)} + \underset{F}{15(2\pi)} \\
&= 2\pi(14-29+15) \\
&= 0.
\end{aligned}$$



sphere (split along arc AB) + Möbius band = projective plane

FIGURE 3

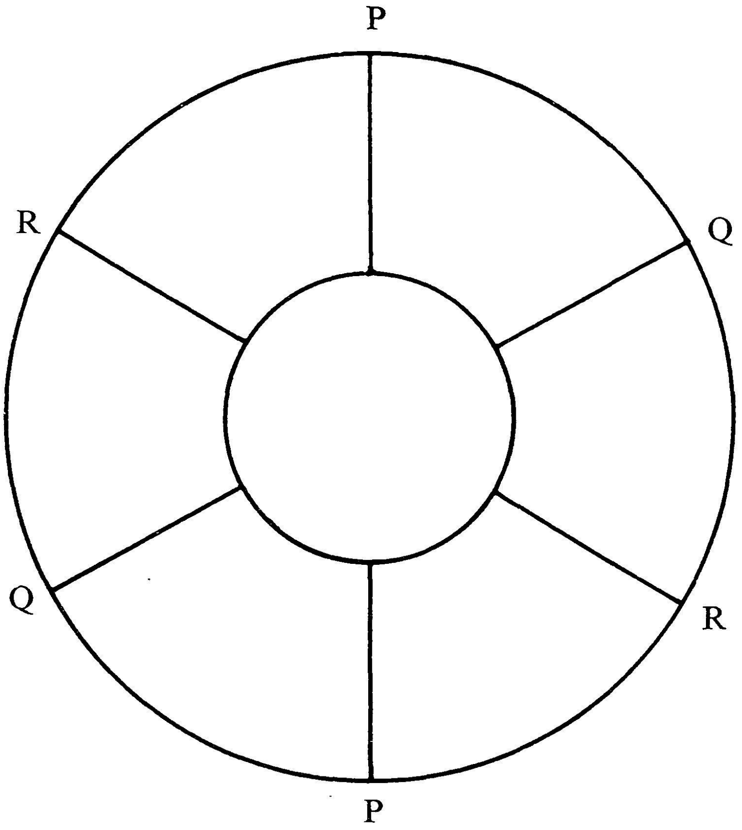


FIGURE 4

A non-orientable surface may be formed from a sphere S^2 by adding cross-caps (i.e., Möbius bands). If one cross-cap is added, we have the projective plane (see Figure 3). In general, for a non-orientable surface S with k cross-caps

$$\chi(S) = 2 - k. \tag{2.5}$$

We now exemplify Theorem 1 for the projective plane ($k=1$). A cellular subdivision of the projective plane is shown in Figure 4 (where, for aesthetic reasons, we have maintained the rounded edges rather than draw, artificially, a strictly polyhedral figure). The cells consist of 6 quadrilaterals and one hexagon, so that the sum of all the face angles may be expressed by $6\,(4-2)\,\pi + 1\,(6-2)\,\pi$. There are 9 vertices, 15 edges and 7 faces. We display the computation for Δ in the same manner as the last example so that it may suggest the general approach.

$$\begin{aligned}
\Delta &= 9\,(2\pi) - \{6\,(4-2)\,\pi + 1\,(6-2)\,\pi\} \\
&= \underset{V}{9\,(2\pi)} - \underset{E}{15\,(2\pi)} + \underset{F}{7\,(2\pi)} \\
&= 2\pi\,(9-15+7) \\
&= 2\pi\,(1) \\
&= 2\pi.
\end{aligned}$$

Theorem 1 exhibits a remarkable fact about the total angular defect of P. For, quite apart from the precise relationship between Δ and χ which it expresses, it shows that $\Delta(P)$ depends only on the topological type of P. It would be remarkable enough that $\Delta(P)$ is independent of the cellular subdivision of P; but, in fact, it does not vary if P is replaced by some other polyhedron homeomorphic to P. Thus $\Delta(P)$ may be said, paradoxically, to be defined by the geometry of P—and to be independent of that geometry! In fact the situation is even more remarkable, since the Euler characteristic is not only a topological invariant but even a *homotopy* invariant; this means that we may deform P continuously without changing $\chi(P)$—and thus without changing $\Delta(P)$.

3. The angular defect in higher dimensions

We look now at the possibility of obtaining a formula for the total angular defect for a polyhedron of arbitrary dimension. We will largely confine attention to *polytopes* (see [3]), that is, homeomorphs of [1] S^{n-1}, for some $n \geqslant 3$. As explained in the Introduction, we will no longer expect to find any significant relationship with the Euler characteristic, so we will concentrate on the question of whether, for such a polytope P, we may obtain a formula for $\Delta(P)$ in terms of V, E and F. Our first result is very general, but will prove to be applicable for certain standard polytopes.

THEOREM 2. *Let P be an arbitrary polyhedron in which every edge is incident with precisely q faces, then*

$$\Delta(P) = \pi(2V - qE + 2F). \tag{3.1}$$

Proof. We have only to make a small modification of Pólya's argument. We proceed as in the proof of Theorem 1 as far as the relation (2.2). But now

$$\sum_{j=1}^{F} S_j = qE,$$

so that (2.2) implies that

$$q\pi E - 2\pi F = 2\pi V - \Delta,$$

from which (3.1) immediately follows.

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[1]) We explain later in the section why it is more convenient to talk of S^{n-1} than of S^n.