SPECKER'S MATHEMATICAL WORK FROM 1949 TO 1979

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by Hao WANG

Ernst Specker was born in Zürich sixty years ago today (on February 11, 1920). He is presently an active professor at E.T.H. where he has been since 1940: as student, graduate student, assistant, Privatdozent, and professor. Apart from one year at the Institute for Advanced Study (1949-50) and two years at Cornell University (1958-59, 1961-62), he has only taken short trips away from Zürich. Such an externally undisturbed academic career is indeed exceptional (even in Europe) and it undoubtedly contributes to the cultivation of Specker's many attractive qualities.

In his charming biography of Bernays (item 32 in the appended list), Specker says that Bernays "was unique in his refusal to judge other people". It is true that Specker does not go to such an extreme. But their close association is not accidental. Specker shares certain qualities with Bernays: no malice, no intrigue, a wide range of sympathies, and probably similar preferences in human relations.

In the autumn of 1950 I came to Zürich to study with Paul Bernays. For the following seven or eight months I got the opportunity to take some delightful long walks with Bernays and enjoy many idle evenings with Specker. It is indeed rare to meet, especially with different cultural traditions, persons as congenial as Bernays and Specker. Afterwards Specker and I vacationed together in 1954 and visited one another briefly in 1955, 1963, 1975, and 1978. Even though the contacts have not been frequent, I have all along felt Specker to be somebody specially close.

Specker was ill for an extended period before completing his formal education. He had the elisure to think over many things. This experience may have helped cultivating his superiority as a person. In terms of traditional Chinese categories, I would say there is a taoist trait in him in the sense of being more detached, less competitive, and more understanding. I believe he has a better sense of what is important in life and arranges his life better than most logicians.

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A remarkable feature of Specker's work is what I might call playfulness and leisureliness. It is original but not pretentious, thorough but not laborious, spontaneous and unmindful of fashion or fad. He enjoys more in striking out on new paths than in building upon a mass of material. He began his work in topology but allowed his competing interest in logic to dominate probably because he found it tedious to keep track of who did what in topology already at that time. I often wonder whether he might not have left set theory later for the same reason. He was the star student at the E.T.H. but was humiliated by his ignorance of quantum mechanics at the final examination for his doctorate. I am, therefore, inclined to conjecture that this episode may have been the remote cause of his turning to the task of interpreting quantum mechanics much later.

In agreeing to review Specker's work so far, I overstretch myself and can only hope to reduce serious errors by consulting with others. Not only am I ignorant of topology and quantum mechanics, but for many years I have allowed my diverse interests to run wild, straying far away from technical logic most of the time. Consequently it will be specially hard for me to place Specker's work in a proper perspective relative to current state of mathematics.

Moreover, since I find people more important than mathematics, I would have liked to know more directly about Specker as teacher, as colleague, and as coauthor. As it is I have only second hand information about his Socratic method of teaching, the affection his colleagues have for him, and the pleasure of doing joint research with him. But it could be argued that as a result my testimony is more disinterested and objective.

I count thirty-two papers of which ten are coauthored (with MacDowell, Erdös, Gaifman, Kochen, Hodes, Strassen, Wick and Lieberherr). A simple-minded classification might give an idea of the range of Specker's mathematical interest (compare list at the end of this paper):

I. Topology: 1, 4, 5.

II. Recursive analysis: 2, 12, 22.

III. Combinatorial set theory: 3, 16, 18.

IV. Type theory: 6, 13, 17.

V. Axiomatic set theory: 8, 9, 29.

VI. Ramsey's theorem: 10, 24.

VII. Arithmetic: 11, 15.

VIII. Logic and quantum mechanics: 14 (and 25), 19, 20, 21.

IX. Algorithms: 23, 25, 27, 38, 30, 31.

X. General: 7, 32.

Roughly speaking, over the last thirty years or so, the first 15 years are devoted to the logician's traditional concern with set theory, analysis, and arithmetic (with the work on topology as a prelude supplying a solid training in ordinary mathematics), while the second 15 years are devoted to the logic of quantum mechanics and the study of algorithms. Rather than enumerating the main theorems in the different articles, I shall select more or less arbitrarily to remark on a few of the articles which are representative and not too unfamiliar to me.

From doctorate to professorship

Specker received his doctorate of mathematics in 1948 with his Promotionsarbeit in topology (published as 1949*a*, 1 in the above list). He completed his Habilitationsschrift in axiomatic set theory in 1951, which was later published as 1954*b* and 1957*a* (8 and 9 in the above list). In these studies he worked closely with H. Hopf and Bernays. Later on October 16, 1976 he gave a lecture (29 in the above list) in München on the occasion of the Ehrenpromotion of Bernays in which he traced the development of axiomatic set theory with special attention to the contributions by Bernays. In his Habilitationsschrift, Specker proves the independence of the axiom of foundation, gives a new proof of the independence of the axiom of choice from the axioms of set theory minus the axiom of foundation, studies several alternatives to the axiom of choice, and sharpens results dealing with the relation between the axiom of choice and the generalized continuum hypothesis.

In 1954 Specker gave his Antrittsvorlesung at the E.T.H., in which he discusses the conceptual foundation of set theory (published as 7 in the above list). This is probably the only publication by Specker which would ordinarily be classified as "philosophical" in the specialized academic sense of the word and, at least in his published work, Specker has so far not returned to philosophical considerations which do not directly suggest some mathematical problems. In his general paper on Bernays (32), he notes that about half of Bernays' papers may be classified as philosophical: this difference between them may be partly due to the different historical periods in which they live.

Enjoyment of interaction

Among Specker's publications, several papers seem to have been stimulated primarily by the enjoyment of personal interaction. Thus the paper 1949*c* dealt with a problem of Sikorski who was visiting Zürich then, while the paper 1964 continued the study to more elaborate cases (3 and 18 in the above list). The papers 1957*b* and 1961*b* (10 and 16 in the above list) seem to belong to the class of papers which are provoked by the infinite supply of problems from Erdös. Paper 11 answers a problem raised by Mostowski. A most obviously playful paper is 1978*b* (30) which gives, for the recognition problem, the generating problem, and the counting problem of the partition of finite sets, algorithms programmable on the "toy" computer HP-25.

Several of these papers contain clever constructions which stimulate extensions and generalizations. For example, the paper 10 gives the Specker graph which shows:

$$\omega^3 \to (2, \omega^3)^2 \text{ and } \omega^3 \not\to (3, \omega^3)^2 .$$

This leads to the function $f(n)$ such that $f(n) < \omega$,

$$\omega^n \to (f(n)-1, \omega^3)^2 \text{ and } \omega^n \not\to (f(n), \omega^3)^2 .$$

Eva Nosal many years later showed that $f(n) = 2^{n-2} + 1$ for $n \geqslant 3$, *J. London Math. Soc. (2), 8* (1974), 306-310.

Topology and recursive analysis

It is interesting to observe that Specker's early papers of 1949 and 1950 have continued to interest mathematicians over the years. For example, the paper 2 gives a bounded increasing recursive sequence of rational numbers that does not converge to a recursive real number. In a recent paper by M. I. Kanovič, such sequences are called Specker sequences, and the complexity of "limit candidates" for a Specker sequence is studied with the result that the larger the complexity of the candidate, the closer it is to the actual (more recursive) limit. The reference is: *Soviet Math. Dokl.*, vol. 15 (1974), No. 1, pp. 299-303.

The three papers on recursive analysis are remarkable in staying away from more controversial issues concerning constructivity. In particular, the first paper is probably the earliest use of recursive functions to elucidate constructive analysis.

Of the three papers on topology, I can out of ignorance have very little to say. The paper 1950*b* was done when Specker visited the Institute for Advanced Study. In it he is able to make interesting contributions to group theory as a topologist. It contains elegant ideas which received further development, for instance, almost two decades later in G. Nöbeling, "Verallgemeinerung eines Satzes von Herrn E. Specker", *Inventiones math.*, *6* (1968), 41-55. Let F be the Abelian group of sequences of integers with $\{a_n\} + \{b_n\} = \{a_n + b_n\}$. Specker shows that every countable subgroup of F is free and that F contains a nonfree subgroup of cardinality aleph-one (hence, F itself is not free). Furthermore, let F_b be the subgroup of F consisting of all bounded sequences of integers. Then it is shown that every subgroup of F_b with cardinality aleph-one is a free group. This last theorem is generalized by Nöbeling to show that for an arbitrary set X (rather than just the set of integers), the group of all bounded sequences from X is free and possesses a characteristic basis.

Specker's interest in the group F and its subgroups is suggested by the problem of determining the algebraic structure of the first cohomogy group of an infinite complex, a problem studied in his dissertation 1949*a*. The paper 1950*a* considers end lattices and introduces what is known as Specker compactization which is investigated extensively, for instance, in Herbert Abels, Specker-Kompaktifizierung von lokal kompakten topologischen gruppen, *Math. Z.*, *135* (1974), 325-361.

An example of beginning with concrete problems

In March 1966, Specker lectured in England on his result that Ramsey's theorem does not hold in recursive set theory. Afterwards, he was persuaded to present it at the Logic Colloquium of 1969 (and publish it as 24 in the above list). More exactly, Specker proves that there is a recursive (Σ_0) partition of the 2-elements sets of natural numbers which possesses no recursively enumerable (Σ_1) infinite sets of indiscernibles and that for every recursive partition there is always a Δ_3 set of indiscernibles. This suggests both a question of more exact answers for the case 2 and a more general question of extending the study from 2 to greater n. In fact C. G. Jockush generalizes and settles both questions shortly afterwards *(Journal of symbolic logic*, vol. 37, 1972, pp. 268-280): for every $n \geqslant 2$ and every recursive partition of n-elements sets of natural numbers, there is some Π_n set of indiscernibles; for every $n \geqslant 2$, there is some recursive partition of n-elements sets (into two classes) such that there is no Σ_n set of indiscernibles. In 1977, results by Kirby and Paris aroused widespread interest because their work yields some more mathematical examples of the incompleteness of Peano arithmetic (see, e.g., the last chapter of *Handbook of mathematical logic*, December 1977). At this juncture, several people observed that Jockusch's generalization of Specker's result actually yields quite directly rather similar results, one form of which says simply that Ramsey's theorem is undecidable in the weak (or predicative) second order extension of Peano arithmetic. (A version of the derivation is reported in my *Popular lectures on mathematical logic* being published in Beijing.)

This is in my opinion an illustration of how a good choice of an apparently isolated concrete problem can relate to more substantial developments in a surprising way. Pursuing this line of thought, I would also like to consider another seemingly small beginning by Specker which has been followed by larger results and new vistas.

Logic and quantum mechanics

In 1960, Specker began his consideration of propositions which are not simultaneously decidable with an ancient story about applying to marry a certain princess. Three boxes A, B, C are each empty or contain a ball. The problem is to guess which box is empty and which is not by selecting to open any two boxes which both are conjectured to be empty or nonempty. The boxes are connected in such a way that one can open any two boxes but then the third can no longer be opened. Moreover, the construction is such that whenever any two boxes are open, exactly one of them is empty. Hence nobody was able to win. Finally, somebody insisted on opening two boxes exactly one of which he conjectured to be empty. As he happened to guess right and the third box could no longer be opened, he won the hand of the princess. In this example, the three propositions "x is empty" ($x=A$, B, C) are not simultaneously decidable.

Specker then offers an improved formulation of the logic of quantum mechanics first considered by G. Birkhoff and J. v. Neumann (*Annales of Math.*, 1936). The logic is "partial" because the conjunction (say) of two propositions not simultaneously decidable has no truth value. He proposes, in contrast to Birkhoff and v. Neumann, that conjunctions, etc. of propositions which are not simultaneously decidable can also be introduced. The problem is then posed: "Can the description of a quantum-mechanical system be so extended with auxiliary (fictitious) propositions that the classical propositional logic holds?" This is answered in the negative by showing that the partial Boolean algebra $B(E^3)$ of the linear subspaces of the three dimensional Hilbert space cannot be imbedded in a Boolean algebra.

These observations in Specker's paper of 1960 were followed by systematic joint work with Kochen. The task of axiomatizing the logic of quantum mechanics is accomplished in 1965*a* and 1965*b* with a calculus of partial propositional functions that is shown to satisfy the natural requirements. They continued with the larger paper 1967*a* in which interesting examples are given to show that certain simple partial Boolean subalgebras of $B(E^3)$ cannot be imbedded in a Boolean (commutative) algebra. This yields an improvement (and correction) of J. v. Neumann's well-known theorem on the non-existence of hidden variables in quantum mechanics (*Mathematical foundations of quantum mechanics*, in German, 1932; English translation, 1955). For example, the result is summarized in E. J. Belinfante's *A survey of hidden variables* under the name "the Kochen-Specker paradox" (see p. 37).

Very recently, Kochen is circulating a typescript entitled *The interpretation of quantum mechanics* (89 pp) in which the negative results with Specker are turned around and expanded in all directions to get a new interpretation of quantum mechanics. The concept of interactive properties is taken seriously and elucidated mathematically. It will be interesting to watch how this will be received by physicists and mathematicians specializing in quantum theory.

I should like to turn to an area outside the mainstream of current logic in which Specker's work occupies a central place. This is the area of trying to strengthen the simple theory of types without going into the transfinite types. The best known proposal is Quine's New Foundations (briefly NF, *Am. math. monthly*, vol. 44, 1937, pp. 70-80) which strikes one as highly artificial. Specker not only proves most surprising results about NF but also gives a much more natural equivalent characterization of NF in terms of typical ambiguity.

Typical ambiguity and model theory

The commonly accepted cumulative or iterative concept of set can be viewed as an extension of the simple theory of types to the transfinite. It is often helpful first to confine attention to this simple theory both for exposition and for finding out new facts. For example, Gödel apparently studied the independence of the axiom of choice and the continuum hypothesis in this framework in the 1940s. A tempting question is to look for other extensions of the simple theory of types.

The family of structures intended by the theory is altogether familiar and natural. Let T_0 be a (nonempty) set; elements of T_0 are elements of type 0. T_1 is the set of subsets of T_0; T_2 is the set of subsets of T_1; and in general T_{n+1} is the set of subsets of T_n. We have variables $x_1^0, x_2^0, \ldots,$ $x_1^1, x_2^1, \ldots,$ etc. and prime formulas of two kinds such as $x_4^3 = x_2^3$ and $x_6^2 \in x_6^3$. In this way the language is determined in the obvious way. The intended structure $(T_0, T_1, \ldots, \in, =)$ has $\in$ and $=$ interpreted in the usual way and T_{k+1} taken as the power set of T_k. The axioms are of two groups and they make up the axiom system T $(n=0, 1, 2, \ldots)$:

T1. Extensionality. $\forall x_1^n (x_1^n \in x_1^{n+1} \leftrightarrow x_1^n \in x_2^{n+1}) \to x^{n+1} = x_2^{n+1}$.

T2. Comprehension. $\exists x_1^{n+1} \forall x_2^n (x_2^n \in x_1^{n+1} \leftrightarrow C(x_2^n))$.

Let F^+ be obtained from F by raising the superscripts of every variable in F by 1. A direct result on the system T is:

Theorem 1. If F is a theorem of T, so is F^+.

The converse is certainly not true. Since T_0 is nonempty, we can easily prove there are at least 2^n objects of type n. E.g., we can prove:

$$\exists x_1^1 \exists x_2^1 (x_1^1 \neq x_2^1),$$

call it S^+. But we cannot prove S in T. Once I suggested an extension N of T to include negative types (*Mind*, vol. 61, 1952, pp. 366-368), with the axioms T1 and T2 reconstrued so that n may also take negative integers as values. It could then be shown that, for every n and every given positive k_0, there are more than k_0 sets of type n. Yet it can also be shown in elementary number theory that N is consistent. Hence, for no fixed type n can one prove in N an axiom of infinity (i.e., there are infinitely many sets of type n).

Specker considers a theory T' obtained from T by adding the rule: if $\vdash S^+$, then $\vdash S$. He shows that T' is consistent and every model of N yields one of T'. The more difficult question is whether the system T^+ obtained from T by adding the axiom (scheme) "$S \leftrightarrow S^+$" is consistent. In the paper 1958 (=13), Specker proves the following theorem:

THEOREM 2. The system NF is consistent if and only if T^+ is.

In this way one gets a more natural characterization of NF in terms of "typical ambiguity", because T^+ may be said to be the result of taking typical ambiguity seriously.

In the paper 1962 (=17), Specker further proves:

THEOREM 3. If T^+ is consistent, then there exists a model $(M_0, M_1, \ldots, \in, =)$ which admits an isomorphism mapping M_k onto M_{k+1}.

In 1969, Ronald Jensen combined Specker's way of constructing models with an interesting use of Ramsey's theorem to get yet another surprising result about NF: If the extensionality axiom is weakened to allow individuals (urelements), then the resulting system NFU can be proved consistent in elementary number theory so that the axiom of infinity is not provable in NFU (*Words and objections*, pp. 278-291).

This contrasts with Specker's result of 1953 (=6):

THEOREM 4. The axiom of choice is refutable in NF and so the axiom of infinity is a theorem of NF.

Some time before this, I had remarked on a possible application of Skolem's theorem on countable models. Since NF is known to have a finite axiomatization (T. Hailperin, *Journal of symbolic logic*, vol. 9, 1945, pp. 1-19), I thought that by applying the axiom of choice, one can introduce in NF a set which essentially enumerates a countable model of NF, so that by the diagonal argument a new set and a contradiction can be derived. But my enumeration used an unstratified formula and I do not know whether one can remedy this by some trick to get an alternative proof of Theorem 4.

In regard to the construction of models, jointly with MacDowell, Specker has proved the following well-known theorem (1961*a* = 15; compare also *Handbook of mathematical logic*, p. 79):

THEOREM 5. To every model M of Peano arithmetic, there is a proper elementary extension N of M such that all elements in $N - M$ are greater than all elements of M.

COMPLEXITY OF ALGORITHMS

In recent years under the leadership of Specker (at the E.T.H.) and Volker Strassen (at the Universität), Zürich has become a center for studies in computational complexity. One result is the volume edited by them with their lucid introduction (1976*a*). The center of interest in this volume is to consider whether each of a wide range of problems requires exponential algorithms or can be done in polynomial time. In particular, there is the famous open problem whether $P = NP$. In the Specker-Strassen volume $P \neq NP$ is called Cook's hypothesis (*Proc. of 3rd ACM Sym. on Theory of Computing*, 1971, pp. 151-158). Specker and Strassen who feel that the hypothesis is plausible present the following considerations. For example, most of the algorithmic problems in classical number theory can be interpreted as decision problems of the NP class and yet so far only special cases of such problems have been solved by special methods which are of the polynomial kind. Moreover, Cook's hypothesis is implied by the "spectrum hypothesis" which says that there is some spectrum whose complement is not a spectrum (the spectrum of a first-order formula F is the set of integers n such that F has an n-membered model).

The paper 1976*b* gives an illustration of the situation that sometimes what seems at first sight to require an exponential algorithm may upon closer analysis be seen to possess a polynomial one. Generalizing a result of M. Hall (1956), Specker gives a polynomial algorithm for finding distinct "independent" representations from a finite number of finite sets. (A set U of subsets of a finite set M is an independence structure over M if each subset of a member of U is a member of U, and whenever A, B belong to U and $|A| = |B| + 1$, there is some c in $A - B$ such that $A \cup \{c\}$ belongs to U. A set of representatives of M is independent if it belongs to U).

Both 1968 and 1976*c* study the question of determining the length of formulas in terms of different primitive connectives for representing each function. Essentially the concern is with Boolean functions. The formulas are built up from 0, 1 and the variables, with Boolean connectives. A central concern is to find "intrinsic properties" of functions which make every representing formula of such a function long. In the 1968 paper one of the early lower bounds in complexity theory is established.

The results of 1968 are illustrated in a familiar manner. Let F_1 be the set of Boolean formulas with negation and conjunction as the only connectives, F_2 uses in addition also biconditional, F_3 extends F_2 by allowing also quantification over Boolean variables. It is proved that for every c, there is a formula G in F_{i+1} such that for every formula H in F_i $(i=1, 2)$ equivalent to G, the following holds:

$$\text{length of } H \geqslant c \cdot (\text{length of } G)\,.$$

The two parts of 1976*c* both study the problem of estimating the value of $L(f)$, giving the length of a shortest formula which represents the Boolean function f. The basic tool is the concept of subfunctions contained in a function. Let f be a Boolean function. Then g is a subfunction of f if it is obtained from f by fixing some subset of the variables of f to constants.

The second half of 1976*c* reformulates the ideas of 1968 and brings out the following corollary for symmetric functions. There is a function $t(n)$, $\lim\big(t(n)/n\big) = \infty$, such that for symmetric functions f of n variables (except 16 simple functions for each n), $L(f) > t(n)$.

Based on the kind of technique introduced in 1968, Fisher, Meyer and Paterson (in paper presented at the 7th ACM Symp. on Theory of Computing, May 1975) have proved lower bounds of up to $n \log n$ for a more restricted class of symmetric functions.

The first half of 1976*c* sharpens a result of E. E. Neciporuk *(Soviet math. dokl.*, vol. 7, 1966, pp. 999-1000) and makes three applications. The main result gives a lower bound to $L(f)$ by counting up subfunctions of f:

Roughly speaking, if f is a Boolean function of m variables, and G is a formula representing f with $L(G)$ defined as the number of occurrences of the m variables, then $L(G) > (\Sigma \log e_i)/\log 5$, where e_i is the number of subfunctions over X_i $(i=1, ..., j$ for some $j)$, and $X_1, ..., X_j$ make up a partition of the m variables of the function f.

Specker's most recent publication is 1979*a* which is apparently still in galley proofs. This relates more directly to the P = NP problem in the central case of the tautology problem. Let F be a formula in the conjunctive normal form (CNF). It is said to be 2-satisfiable if any two clauses are simultaneously satisfiable. For example, $p, q, \bar{p} \vee \bar{q}, p \vee q$ is 2-satisfiable but not satisfiable. Let h be the "golden ratio" $(\sqrt{5}-1)/2 \doteq 0.618$, which is the positive solution of $h^2 + h - 1 = 0$.

It is shown that for 2-satisfiable F in CNF, there exists a satisfiable subset of the clauses $C_1, \ldots, C_n$ in F which has hn members. Moreover, there is a polynomial algorithm to find such a set. On the other hand, for any $h' > h$, there is some 2-satisfiable F which contains no satisfiable subset of at least $h' \mid F \mid$ members ($\mid F \mid$ being the number of clauses in F).

Let $Z(a)$ be the set of CNF's such that each F in CNF has an interpretation satisfying $a \mid F \mid$ clauses. The construction problem of $Z(a)$ is to compute for each F in $Z(a)$ an interpretation which satisfies at least $a \mid F \mid$ clauses. In this terminology it is well-known that P = NP iff the construction problem of $Z(1)$ is in P. The result mentioned above shows that the construction problem for 2-satisfiable CNF's in $Z(h)$ is in P. Let now h' be an algebraic number such that $1 \geqslant h' > h$. A somewhat mysterious result is then given: the construction problem for all 2-satisfiable CNF's in $Z(h')$ is in P, iff P = NP. In other words, the set of 2-satisfiable CNF's which belong to $Z(h')$ is NP-complete.

Specker and his coauthor remark that under Cook's hypothesis (i.e., $P \neq$ NP), there is a "quantum jump" at h, because at this point, the complexity of computation passes over from P to NP which is no longer polynomial under Cook's hypothesis. They do not mention whether they consider their result to be positive or negative evidence for Cook's conjecture. Over the years I have asked several experts why they believe in the conjecture and have failed to be convinced by the reasons they give. I continue to feel that our state of ignorance today is such that nothing is known to make $P \neq$ NP seem more plausible than P = NP.

According to Specker, the most important implication of 1979*a* is to draw attention to the golden ratio: we should not expect to fulfill more than 61.8% of our wishes.

Specker's Mathematical Publications (1949-79)

1. 1949*a*. Die erste Cohomologiegruppe von Überlagerungen und Homotopieeigenschaften dreidimonsionaler Mannigfaltigkeiten. *Commentarii Mathematici Helvetici*, vol. 23, pp. 303-333. Promotionsarbeit for Doctor of Mathematics at ETH, June, 1948.
2. 1949*b*. Nicht konstruktiv beweisbare Sätze der Analysis. *Journal of symbolic logic*, vol. 14, pp. 145-158.
3. 1949*c*. Sur un problème de Sikorski. *Colloquium Mathematicum*, vol. 2, pp. 9-12.
4. 1950*a*. Endenverbände von Räumen und Gruppen. *Math. Annalen*, vol. 122, pp. 167-174.

5. 1950*b*. Additive Gruppen von Folgen ganzer Zahlen. *Portugaliae Mathematica*, vol. 9, pp. 131-140.
6. 1953. The axiom of choice in Quine's new foundations for mathematical logic. *Proc. Nat. Acad. Sci. U.S.A.*, vol. 39, pp. 972-975.
7. 1954*a*. Die antinomie der Mengenlehre. *Dialectica*. vol. 8, pp. 234-244. Antrittsvorlesung at the ETH.
8. 1954*b*. Verallgemeinerte Kontinuumshypothese und Auswahlaxiom. *Archiv der Mathematik*, vol. 5, pp. 332-337.
9. 1957*a*. Zur Axiomatik der Mengenlehre (Fundierungs- und Auswahlaxiom). *Zeitschr. f. math. Logik und Grundlagen d. Math.*, vol. 3, pp. 173-210. This and 1954*b* make up the 1951 Habilitationsschrift at ETH.
10. 1957*b*. Teilmengen von Mengen mit Relationen. *Commentarii Mathematici Helvetici*, vol. 31, pp. 302-314.
11. 1957*c*. Eine Verschärfung des Unvollständigkeitssatzes der Zahlentheorie. *Bull. Acad. Polonais des Sciences, cl. III*, vol. 5, pp. 1041-1045.
12. 1957*d*. Der Satz vom Maximum in der rekursiven Analysis. *Constructivity in mathematics* (Proc. of 1957 colloquium at Amsterdam), pp. 254-265.
13. 1958. Dualität. *Dialectica*, vol. 12, pp. 451-465.
14. 1960. Die Logik nicht gleizeitig entscheidbarer Aussagen. *Dialectica*, vol. 14, pp. 239-246.
15. 1961*a*. (With R. MacDowell). Modelle der Arithmetik. *Infinitistic Methods*, Pergamon, London, pp. 257-263.
16. 1961*b*. (With P. Erdös). On a theorem in the theory of relations and a solution of a problem of Knaster. *Colloquium Mathematicum*, vol. 8, pp. 1921.
17. 1962. Typical ambiguity. *Logic, methodology and philos. of sci.: Proc. of the 1960 Congress*, pp. 116-124.
18. 1964. (With Haim Gaifman). Isomorphism types of trees. *Proc. Am. Math. Soc.*, vol. 15, pp. 1-7.
19. 1965*a*. (With Simon Kochen). Logical structures arising in quantum theory. *Symposium on the theory of models*, Amsterdam, pp. 177-189.
20. 1965*b*. (With Simon Kochen). The calculus of partial propositional functions. *Proc. 1964 Congress for Logic, Methodology and Philos. of Sci.*, pp. 45-57.
21. 1967*a*. (With Simon Kochen). The problem of hidden variables in quantum mechanics. *J. math. and mechanics*, vol. 17, pp. 59-88.
22. 1967*b*. The fundamental theorem of algebra in recursive analysis. *Proc. Symposium Zürich-Ruschlikon*, pp. 321-329.
23. 1968. (With L. Hodes). Length of formulas and elimination of quantifiers I. *Contributions to math. logic*, Amsterdam, ed. K. Schütte, pp. 175-188.
24. 1969. Ramsey's theorem does not hold in recursive set theory. *Logic Colloquium '69*, pp. 439-442. The result of this paper has been presented at the Universities of Bristol and Manchester in March 1966.
25. 1975. Logic of propositions not simultaneously decidable. *The logico-algebraic approach to quantum mechanics*, ed. C. A. Hooker, pp. 135-140. (This is an English translation of item 14.)
26. 1976*a*. (With Volker Strassen). Einleitung, *Komplexität von Entscheidungsproblemen*, pp. 1-10. They are also the editors of the volume.
27. 1976*b*. Ein polynomialer Algorithmus zur Bestimmung unabhängiger Repräsentantensystem, *ibid.*, pp. 72-85.
28. 1976*c*. (With G. Wick). Länge und Formeln, *ibid.*, pp. 182-217.
29. 1978*a*. Die Entwicklung der axiomatische Mengenlehre. *Jber. d. Dt. Math.-Verein.*, vol. 81, pp. 13-21.

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