

## 2. Some notions from logic

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **27 (1981)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **25.05.2024**

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space upper bound for the  $\exists^* \forall \exists$  subcase is obtained at the same time. It is easy to see that the class  $\exists^* \forall$  is *NP*-complete.

Section six contains the main result, namely the  $c^{n/\log n}$  lower bound for the  $\forall \exists \exists$  case, and also a tight lower bound for the  $\forall \exists$  case, as well as some *NP*-complete problems. In the last section are some conclusions.

## 2. SOME NOTIONS FROM LOGIC

The formulas of first order logic (see e.g. Shoenfield [36]) are built from:

- variables  $y, x_1, x_2, \dots, z_1, z_2, \dots$
- function symbols  $f, g, f_L, f_R, f_1, f_2, \dots$   
(we use  $c, c_1, c_2, \dots$  for 0-any function symbols, i.e. constants)
- predicate symbols  $P, P_1, P_2, \dots$  (and other capitals)
- auxiliary symbols  $(, )$
- equality symbol  $=$
- propositional symbols  $\wedge, \vee, \neg, \rightarrow, \leftrightarrow$
- quantifiers  $\forall, \exists$

We use  $F[x/t]$  to denote the result of the *substitution* of the term  $t$  for the variable  $x$  in the formula  $F$ .

A formula  $Q_1 x_1 Q_2 x_2 \dots Q_n x_n F_0$  with  $Q_i$  quantifiers (for  $i = 1, \dots, n$ ) and  $F_0$  quantifier-free is in *prenex form*.  $F_0$  is called the *matrix* of the formula.

We are investigating decision procedures for formulas of first order logic without equality and without function symbols. But we use the functional form of formulas.

The *functional form* of a formula in prenex form is constructed by repeatedly changing

$$\forall x_1 \forall x_2 \dots \forall x_n \exists y F \quad (F \text{ may contain quantifiers}) \text{ to}$$

$$\forall x_1 \forall x_2 \dots \forall x_n F[y/f_i(x_1, \dots, x_n)]$$

using each time a new  $n$ -ary function symbol  $f_i$  until no more existential quantifiers appear.

A formula is satisfiable, iff its functional form is satisfiable. In addition, both are satisfiable by structures of the same cardinality.

We use  $\alpha, \alpha'$  to denote structures. A *structure*  $\alpha$  for a first order language  $L$  consists of:

- a nonempty set  $|\alpha|$  (the universe of  $\alpha$ ),
- a function  $f^\alpha : |\alpha|^n \rightarrow |\alpha|$  for each  $n$ -ary function symbol  $f$  of  $L$ , (in particular an individual (= element)  $c^\alpha$  of  $|\alpha|$  for each constant  $c$  of  $L$ ),
- a predicate  $P^\alpha : |\alpha|^n \rightarrow \{\text{true, false}\}$  for each  $n$ -ary predicate symbol  $P$  in  $L$ .

$f^\alpha$  and  $P^\alpha$  are called interpretations of  $f$  and  $P$ .

A structure for a language  $L$  defines a truth-value for each closed formula (i.e. formula without free variables) of  $L$  in the obvious way (see e.g. [36]). A structure  $\alpha$  is a *model* of a set of closed formulas, if all the formulas of the set get the value true (i.e. are *valid* in  $\alpha$ ). A formula  $F$  is *satisfiable*, if its negation  $\neg F$  is not valid.

Let  $\alpha$  be the following structure for a language  $L$  without equality:

The universe  $|\alpha|$  (the Herbrand universe) is the set of terms built with the function symbols of  $L$  (resp. of  $L$  together with the constant  $c$ , if  $L$  contains no constants (= 0-ary function symbols)). Each function symbol  $f$  is interpreted by  $f^\alpha$  with the property: For each term  $t$ ,  $f^\alpha(t)$  is the term  $f(t)$ . We call such an  $\alpha$  a Herbrand structure. If a formula  $F$  (in the language  $L$ ) is valid in  $\alpha$ , then we call  $\alpha$  a *Herbrand model* of  $F$ .

The following version of the Löwenheim Skolem theorem is very useful for our investigations.

**THEOREM.** *The functional form of a closed formula without equality is satisfiable iff it has a Herbrand model.* □

This theorem can be proved with the methods developed by Löwenheim [29] and completed as well as simplified by Skolem [38]. The version of Skolem [37] which uses the axiom of choice, has less connections with this theorem. Also in Ackermann [2] and Büchi [8] versions of the above theorem are present. Probably for the first time, Ackermann [1] constructs a kind of Herbrand model, the other authors use natural numbers instead.

### 3. SOME NOTIONS FROM COMPUTATIONAL COMPLEXITY

We use one-tape Turing machines and multi-tape Turing machines with a two-way read-only input tape and, if necessary, a one-way write-only output tape. The other tapes are called work tapes. The Turing machine