

# §4. The degree of the canonical line bundle

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*simple pole* and no other singularities. It is easy to see that  $f: X \rightarrow \mathbf{P}^1$  is then an isomorphism.

(3.9) COROLLARY. *If  $D \sim D'$ , then  $\deg D = \deg D'$ .*

*Proof:*  $D \sim D'$  implies  $\mathcal{O}(D) \approx \mathcal{O}(D')$ , hence  $\chi(\mathcal{O}(D)) = \chi(\mathcal{O}(D'))$ . Hence  $\deg D = \deg D'$  by (3.4).

(3.10) *Definition.* The *degree* of a line bundle  $\mathcal{L}$  is the degree of any  $D \in \text{Div } X$  such that  $\mathcal{L} \approx \mathcal{O}(D)$ , i.e. the degree of the divisor of any meromorphic section of  $\mathcal{L}$ .

(3.11) *Remark.* The above definition is justified by (2.11) and (3.9). It is clear that the map  $\deg : \text{Pic } X \rightarrow \mathbf{Z}$  is a group homomorphism.

(3.13) *Definition.* The *degree* of a vector bundle  $\mathcal{V}$  is that of the line bundle  $\det \mathcal{V} = \bigwedge^r \mathcal{O}_x \mathcal{V}, r = \text{rank } \mathcal{V}$ .

(3.14) *Remark.* The stalk of  $(\det \mathcal{V})^{-1} = \text{Hom}(\det \mathcal{V}, \mathcal{O}_x)$  at any  $P \in X$  consists  $\mathcal{O}_P$ -multilinear alternate maps  $\mathcal{V}_P \times \dots \times \mathcal{V}_P$  ( $r$  times)  $\rightarrow \mathcal{O}_P$ .

(3.15) PROPOSITION. *If  $0 \rightarrow \mathcal{V}' \rightarrow \mathcal{V} \rightarrow \mathcal{V}'' \rightarrow 0$  is an exact sequence of vector bundles, then  $\deg \mathcal{V} = \deg \mathcal{V}' + \deg \mathcal{V}''$ .*

*Proof:*  $\det \mathcal{V} \approx \det \mathcal{V}' \otimes \det \mathcal{V}''$ .

(3.16) PROPOSITION. (Riemann-Roch theorem, preliminary form). *For any vector bundle  $\mathcal{V}$  on  $X$ ,*

$$\chi(\mathcal{V}) = \deg \mathcal{V} + \text{rank } \mathcal{V} \cdot \chi(\mathcal{O})$$

*Proof:* In view of (3.15), (3.2) and (2.11), the proposition follows from (3.4) by induction on rank  $\mathcal{V}$ .

#### § 4. THE DEGREE OF THE CANONICAL LINE BUNDLE

Recall that the canonical line bundle  $K$  on  $X$  is the sheaf of holomorphic 1-forms.

(4.1) THEOREM.  $\deg K = 2g - 2 = -2\chi(\mathcal{O})$ .

*Proof:* Choose any nonconstant meromorphic function for  $X$ , and consider the holomorphic map  $f : X \rightarrow \mathbf{P}^1 = Y$ . Then the *ramification divisor*  $R = \sum e(P)P$  of  $f$  is defined as follows: for suitable uniformising parameters  $z$  and  $w$  at  $P$  and  $f(P)$  respectively,  $w(f(z)) = z^{e(P)+1}$ . After composing  $f$  with a fractional linear transformation if necessary, we may assume that  $f$  is unramified over  $\infty$ , i.e.  $e(P) = 0$  if  $f(P) = \infty$ . Note that  $r = \sum_{P \in f^{-1}(Q)} (e(P)+1)$  is independent of  $Q \in Y$ , being clearly the rank of the vector bundle  $f_0(\mathcal{O}_X)$  on  $Y$  (cf. (2.4)). Now  $df$  is a meromorphic 1-form on  $X$  (i.e. a meromorphic section of  $K_X$ ), with zeros of orders  $e(P)$  at the  $P$  with  $f(P) \neq \infty$ , and poles of order two at each of the  $r$  poles of  $f$ . Thus we have:

$$(4.2) \quad (\text{Riemann-Hurwitz formula}). \quad \deg K = \deg R - 2r.$$

On the other hand, by (2.16) and (3.16), we have

$$(4.3) \quad \begin{aligned} \chi(\mathcal{O}_X) &= \chi(f_0(\mathcal{O}_X)) = \deg f_0(\mathcal{O}_X) + r \chi(\mathcal{O}_Y) \\ &= \deg f_0(\mathcal{O}_X) + r. \end{aligned}$$

Thus, to finish the proof of (4.1), we must prove:

$$(4.4) \quad \deg f_0(\mathcal{O}_X) = -\frac{1}{2} \deg R.$$

To prove (4.4), let  $\mathcal{L} = \det f_0(\mathcal{O}_X)$ . Then we shall show that there is a canonical  $\mathcal{O}_Y$ -linear map  $\delta : \mathcal{L} \otimes \mathcal{L} \rightarrow \mathcal{O}_Y$  which, at any  $Q \in Y$ , looks like multiplication by  $t_Q^{\delta(Q)}$ , where  $\delta(Q) = \sum_{P \in f^{-1}(Q)} e(P)$  ( $t_Q$  a uniformising parameter at  $Q$ ). Since  $\sum_Q \delta(Q) = \deg R$ , this will prove (4.4).

The map  $\delta$  is the classical discriminant map. To define it, we first define the “trace” map  $\tau : f_0(\mathcal{O}_X) \rightarrow \mathcal{O}_Y$ : for  $U \subset Y$  open and  $h \in \mathcal{O}_X(f^{-1}(U))$ ,  $\tau(h)(Q) = \sum_{P \in f^{-1}(Q)} (e(P)+1) h(P)$  for all  $Q \in U$ . Then clearly  $\tau(h) \in \mathcal{O}_Y(U)$ , and  $\tau$  is  $\mathcal{O}_Y$ -linear. Now for any  $U \subset Y$  open and any two  $r$ -tuples  $\lambda = (\lambda_1, \dots, \lambda_r), \mu = (\mu_1, \dots, \mu_r)$  of elements of  $\mathcal{O}_X(f^{-1}(U))$  (recall that  $r = \text{rank } f_0(\mathcal{O}_X)$ ), we set  $\delta(\lambda, \mu) = \det(\tau(\lambda_i \mu_j))$ . Clearly  $\delta$  is  $\mathcal{O}_Y$ -multilinear and alternating in each of  $\lambda$  and  $\mu$ , hence defines an  $\mathcal{O}_X$ -linear map

$$\delta : \mathcal{L} \otimes \mathcal{L} \rightarrow \mathcal{O}_Y$$

This is the desired map. To compute the effect of  $\delta$  at any  $Q \in Y$ , let us assume first that  $f^{-1}(Q)$  is a single point  $P$ . In suitable coordinate

systems at  $P$  and  $Q$ ,  $f$  is the map  $Z \rightarrow Z^{e_p+1} = w$  of the unit disc  $U \subset \mathbf{C}$  onto another copy  $W$  of it. Since  $1, Z, \dots, Z^{e_p}$  provide an  $\mathcal{O}_w$ -basis for  $f_0(Q_U)$ , the value of  $\delta$  on a local generator of  $\mathcal{L} \otimes \mathcal{L}$  is given by

$$\det(\tau(Z^{i+j})) , \quad 0 \leq i, j \leq e = e_p .$$

But

$$\tau(Z^{i+j}) = Z^{i+j} (1 + \zeta^{i+j} + (\zeta^{i+j})^2 + \dots + (\zeta^{i+j})^e) ,$$

( $\zeta$  denoting a primitive  $(e+1)-st$  root of unity), hence

$$\begin{aligned} \tau(Z^{i+j}) &= (e+1)Z^{i+j} \quad \text{if } i+j = 0 \quad \text{or} \quad e+1 , \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

Hence  $\det(\tau(Z^{i+j}))$  is a (nonzero) constant multiple of  $Z^{(e+1)e} = w^e$  as asserted.

If  $f^{-1}(Q)$  consists of several points, the situation is a direct sum of those considered above, and  $\delta$  is indeed as asserted. This proves Theorem (4.1).

(4.5) *Remark.* Let the notation be as above, and let  $E(X)$  denote the topological Euler-Poincaré characteristic of  $X$ . Then, using the formula  $E(X) = \text{number of vertices} - \text{number of edges} + \text{number of faces}$  in any triangulation of  $X$ , it is easy to see that  $E(X) = rE(Y) - \deg R(Y = \mathbf{P}^1)$ . Indeed, choose any triangulation of  $Y$  which contains all the images of the ramification points of  $f$  as vertices, and lift it to a triangulation of  $X$ . Then, while  $r$  edges or faces lie over each edge or face of  $Y$ , the ramification points reduce the number of vertices over certain vertices of  $Y$ , and one gets the formula asserted. Since  $E(Y) = 2$ , (4.2) yields:

(4.6) COROLLARY.  $\deg K_X = -E(X) = 2g - 2$ , i.e.  $g$  is also the topological genus  $(1/2)b_1(X)$  of the compact oriented surface  $X$ .

## § 5. RIEMANN-ROCH THEOREM (FINAL FORM). SERRE DUALITY

(5.1) (RIEMANN-ROCH THEOREM). For any line bundle  $\mathcal{L}$  on  $X$ ,

$$h^0(\mathcal{L}) - h^0(K \otimes \mathcal{L}^{-1}) = \deg \mathcal{L} - g + 1 .$$

*Proof:* It is enough to prove