§4. The degree of the canonical line bundle

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simple pole and no other singularities. It is easy to see that $f : X \to \mathbf{P}^1$ is then an isomorphism.

(3.9) COROLLARY. *If* $D \sim D'$, *then* $\deg D = \deg D'$.

Proof: $D \sim D'$ implies $\mathcal{O}(D) \approx \mathcal{O}(D')$, hence $\chi(\mathcal{O}(D)) = \chi(\mathcal{O}(D'))$. Hence $\deg D = \deg D'$ by (3.4).

(3.10) *Definition.* The *degree* of a line bundle $\mathcal{L}$ is the degree of any $D \in \operatorname{Div} X$ such that $\mathcal{L} \approx \mathcal{O}(D)$, i.e. the degree of the divisor of any meromorphic section of $\mathcal{L}$.

(3.11) *Remark.* The above definition is justified by (2.11) and (3.9). It is clear that the map $\deg : \operatorname{Pic} X \to \mathbf{Z}$ is a group homomorphism.

(3.13) *Definition.* The *degree* of a vector bundle $\mathcal{V}$ is that of the line bundle $\det \mathcal{V} = \overset{r}{\wedge}_{\mathcal{O}_X} \mathcal{V}$, $r = \operatorname{rank} \mathcal{V}$.

(3.14) *Remark.* The stalk of $(\det \mathcal{V})^{-1} = \operatorname{Hom}(\det \mathcal{V}, \mathcal{O}_X)$ at any $P \in X$ consists $\mathcal{O}_P$-multilinear alternate maps $\mathcal{V}_P \times \ldots \times \mathcal{V}_P$ (r times) $\to \mathcal{O}_P$.

(3.15) PROPOSITION. *If* $0 \to \mathcal{V}' \to \mathcal{V} \to \mathcal{V}'' \to 0$ *is an exact sequence of vector bundles, then* $\deg \mathcal{V} = \deg \mathcal{V}' + \deg \mathcal{V}''$.

Proof: $\det \mathcal{V} \approx \det \mathcal{V}' \otimes \det \mathcal{V}''$.

(3.16) PROPOSITION. (Riemann-Roch theorem, preliminary form). *For any vector bundle* $\mathcal{V}$ *on* X,

$$\chi(\mathcal{V}) = \deg \mathcal{V} + \operatorname{rank} \mathcal{V} \,.\, \chi(\mathcal{O})$$

Proof: In view of (3.15), (3.2) and (2.11), the proposition follows from (3.4) by induction on rank $\mathcal{V}$.

§ 4. THE DEGREE OF THE CANONICAL LINE BUNDLE

Recall that the canonical line bundle K on X is the sheaf of holomorphic 1-forms.

(4.1) THEOREM. $\deg K = 2g - 2 = -2\,\chi(\mathcal{O})$.

Proof: Choose any nonconstant meromorphic function for X, and consider the holomorphic map $f: X \to \mathbf{P}^1 = Y$. Then the *ramification divisor* $R = \Sigma\, e(P)\, P$ of f is defined as follows: for suitable uniformising parameters z and w at P and $f(P)$ respectively, $w\big(f(z)\big) = z^{e(P)+1}$. After composing f with a fractional linear transformation if necessary, we may assume that f is unramified over ∞, i.e. $e(P) = 0$ if $f(P) = \infty$. Note that $r = \sum_{P \in f^{-1}(Q)} \big(e(P)+1\big)$ is independent of $Q \in Y$, being clearly the rank of the vector bundle $f_0(\mathcal{O}_X)$ on Y (cf. (2.4)). Now df is a meromorphic 1-form on X (i.e. a meromorphic section of K_X), with zeros of orders $e(P)$ at the P with $f(P) \neq \infty$, and poles of order two at each of the r poles of f. Thus we have:

(4.2) *(Riemann-Hurwitz formula)*. $\deg K = \deg R - 2r$.

On the other hand, by (2.16) and (3.16), we have

$$\chi(\mathcal{O}_X) = \chi\big(f_0(\mathcal{O}_X)\big) = \deg f_0(\mathcal{O}_X) + r\,\chi(\mathcal{O}_Y) \tag{4.3}$$
$$= \deg f_0(\mathcal{O}_X) + r\,.$$

Thus, to finish the proof of (4.1), we must prove:

$$\deg f_0(\mathcal{O}_X) = -\frac{1}{2}\deg R\,. \tag{4.4}$$

To prove (4.4), let $\mathscr{L} = \det f_0(\mathcal{O}_X)$. Then we shall show that there is a canonical $\mathcal{O}_Y$-linear map $\delta : \mathscr{L} \otimes \mathscr{L} \to \mathcal{O}_Y$ which, at any $Q \in Y$, looks like multiplication by $t_Q^{\delta(Q)}$, where $\delta(Q) = \sum_{P \in f^{-1}(Q)} e(P)$ (t_Q a uniformising parameter at Q). Since $\sum_Q \delta(Q) = \deg R$, this will prove (4.4).

The map δ is the classical discriminant map. To define it, we first define the "trace" map $\tau : f_0(\mathcal{O}_X) \to \mathcal{O}_Y$: for $U \subset Y$ open and $h \in \mathcal{O}_X\big(f^{-1}(U)\big)$, $\tau(h)(Q) = \sum_{P \in f^{-1}(Q)} \big(e(P)+1\big)\, h(P)$ for all $Q \in U$. Then clearly $\tau(h) \in \mathcal{O}_Y(U)$, and τ is $\mathcal{O}_Y$-linear. Now for any $U \subset Y$ open and any two r-tuples $\lambda = (\lambda_1, \ldots, \lambda_r)$, $\mu = (\mu_1, \ldots, \mu_r)$ of elements of $\mathcal{O}_X\big(f^{-1}(U)\big)$ (recall that $r = \operatorname{rank} f_0(\mathcal{O}_X)$), we set $\delta(\lambda, \mu) = \det\big(\tau(\lambda_i \mu_j)\big)$. Clearly δ is $\mathcal{O}_Y$-multilinear and alternating in each of λ and μ, hence defines an $\mathcal{O}_X$-linear map

$$\delta : \mathscr{L} \otimes \mathscr{L} \to \mathcal{O}_Y$$

This is the desired map. To compute the effect of δ at any $Q \in Y$, let us assume first that $f^{-1}(Q)$ is a single point P. In suitable coordinate systems at P and Q, f is the map $Z \to Z^{e_P+1} = w$ of the unit disc $U \subset \mathbf{C}$ onto another copy W of it. Since $1, Z, \ldots, Z^{e_P}$ provide an $\mathcal{O}_W$-basis for $f_0(Q_U)$, the value of δ on a local generator of $\mathcal{L} \otimes \mathcal{L}$ is given by

$$\det\left(\tau\left(Z^{i+j}\right)\right), \ 0 \leqslant i, j \leqslant e = e_p .$$

But

$$\tau\left(Z^{i+j}\right) = Z^{i+j}\left(1 + \zeta^{i+j} + \left(\zeta^{i+j}\right)^2 + \ldots + \left(\zeta^{i+j}\right)^e\right),$$

(ζ denoting a primitive $(e+1) - st$ root of unity), hence

$$\begin{aligned} \tau\left(Z^{i+j}\right) &= (e+1)\, Z^{i+j} \quad \text{if} \quad i+j = 0 \quad \text{or} \quad e+1, \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

Hence $\det\left(\tau\left(Z^{i+j}\right)\right)$ is a (nonzero) constant multiple of $Z^{(e+1)e} = w^e$ as asserted.

If $f^{-1}(Q)$ consists of several points, the situation is a direct sum of those considered above, and δ is indeed as asserted. This proves Theorem (4.1).

(4.5) *Remark.* Let the notation be as above, and let $E(X)$ denote the topological Euler-Poincaré characteristic of X. Then, using the formula $E(X) =$ number of vertices $-$ number of edges $+$ number of faces in any triangulation of X, it is easy to see that $E(X) = r\,E(Y) - \deg R\,(Y = \mathbf{P}^1)$. Indeed, choose any triangulation of Y which contains all the images of the ramification points of f as vertices, and lift it to a triangulation of X. Then, while r edges or faces lie over each edge or face of Y, the ramification points reduce the number of vertices over certain vertices of Y, and one gets the formula asserted. Since $E(Y) = 2$, (4.2) yields:

(4.6) COROLLARY. $\deg K_X = -E(X) = 2g - 2$, *i.e.* g *is also the topological genus* $(1/2)\, b_1(X)$ *of the compact oriented surface* X.

§ 5. RIEMANN-ROCH THEOREM (FINAL FORM). SERRE DUALITY

(5.1) (RIEMANN-ROCH THEOREM). For any line bundle $\mathcal{L}$ on X,

$$h^0(\mathcal{L}) - h^0(K \otimes \mathcal{L}^{-1}) = \deg \mathcal{L} - g + 1 .$$

Proof: It is enough to prove