

4.1. Naimark equivalence

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **28 (1982)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **26.05.2024**

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

3.3.5. Further applications of the irreducibility criterium in Theorem 3.2 can be found in MILLER [32, Lemmas 3.2 and 4.5] for the Euclidean motion group of \mathbf{R}^2 and for the harmonic oscillator group, TAKAHASHI [39, §3.4] for the discrete series of $SL(2, \mathbf{R})$ and [41, p. 560, Cor. 2] for the spherical principal series of $F_{4(-20)}$.

3.3.6. The method of this section does not show in an *a priori* way that a K -multiplicity free principal series representation has only finitely many irreducible subquotient representations. Actually, this property holds quite generally, cf. WALLACH [45, Theorem 8.13.3].

4. EQUIVALENCES BETWEEN IRREDUCIBLE SUBQUOTIENT REPRESENTATIONS OF THE PRINCIPAL SERIES

4.1. NAIMARK EQUIVALENCE

In this subsection we derive a criterium (Theorem 4.5) for Naimark equivalence of K -multiplicity free representations. Lemmas 4.3 and 4.4 are preparations for its proof.

Let G be an lcsc. group.

Definition 4.1. Let σ and τ be Hilbert representations of G . The representation σ is called *Naimark related* to τ if there is a closed (possibly unbounded) injective linear operator A from $\mathcal{H}(\sigma)$ to $\mathcal{H}(\tau)$ with domain $\mathcal{D}(A)$ dense in $\mathcal{H}(\sigma)$ and range $\mathcal{R}(A)$ dense in $\mathcal{H}(\tau)$ such that $\mathcal{D}(A)$ is σ -invariant and $A\sigma(g)v = \tau(G)Av$ for all $v \in \mathcal{D}(A)$, $g \in G$. Then we use the notation $\sigma \simeq \tau$ or $\overset{A}{\sigma} \simeq \tau$.

Naimark relatedness is not necessarily a transitive relation (cf. WARNER [48, p. 242]). However, we will see that it becomes an equivalence relation (called *Naimark equivalence*) when restricted to the class of unitary representations or of K -multiplicity free representations, K abelian.

Two unitary representations σ and τ of G are called *unitarily equivalent* if there is an isometry A from $\mathcal{H}(\sigma)$ onto $\mathcal{H}(\tau)$ such that $A\sigma(g)v = \tau(g)Av$ for all $v \in \mathcal{H}(\sigma)$, $g \in G$. Clearly unitary equivalence is an equivalence relation.

PROPOSITION 4.2. *Two unitary representations of an lcsc. group G are Naimark related if and only if they are unitarily equivalent.*

See WARNER [48, Prop. 4.3.1.4] for the proof.

Let K be a compact abelian subgroup of G . Let σ and τ be K -multiplicity free representations of G . Let $\{\phi_\delta\}$ and $\{\psi_\delta\}$ be K -bases for $\mathcal{H}(\sigma)$ and $\mathcal{H}(\tau)$, respectively.

LEMMA 4.3. *If $\sigma \xrightarrow{A} \tau$ then $\mathcal{M}(\sigma) = \mathcal{M}(\tau)$, $\phi_\delta \in \mathcal{D}(A)$ and $\psi_\delta \in \mathcal{R}(A)$ ($\delta \in \mathcal{M}(\sigma)$), and there are nonzero complex numbers c_δ ($\delta \in \mathcal{M}(\sigma)$) such that*

$$(4.1) \quad (Av, \psi_\delta) = c_\delta(v, \phi_\delta), \quad v \in \mathcal{D}(A).$$

In particular

$$(4.2) \quad A\phi_\delta = c_\delta\psi_\delta.$$

Proof. Let $\delta \in \mathcal{M}(\sigma)$. Let $v \in \mathcal{D}(A)$. We have, by the intertwining property of A ,

$$\begin{aligned} \int_K \delta(k^{-1})\sigma(k)vdk &= (v, \phi_\delta)\phi_\delta, \\ \int_K \delta(k^{-1})A\sigma(k)vdk &= \int_K \delta(k^{-1})\sigma(k)Avdk \\ &= \begin{cases} (Av, \psi_\delta)\psi_\delta & \text{if } \delta \in \mathcal{M}(\tau), \\ 0 & \text{if } \delta \notin \mathcal{M}(\tau). \end{cases} \end{aligned}$$

Since A is closed, we conclude that $(v, \phi_\delta)\phi_\delta \in \mathcal{D}(A)$ and

$$A((v, \phi_\delta)\phi_\delta) = \begin{cases} (Av, \psi_\delta)\psi_\delta & \text{if } \delta \in \mathcal{M}(\tau), \\ 0 & \text{if } \delta \notin \mathcal{M}(\tau). \end{cases}$$

Since A is injective with dense domain, the left hand side is nonzero for certain $v \in \mathcal{D}(A)$. Hence $\delta \in \mathcal{M}(\tau)$, $\phi_\delta \in \mathcal{D}(A)$ and (4.2) and (4.1) hold for certain nonzero c_δ . Finally, since A is closed with dense range, $\mathcal{M}(\sigma) = \mathcal{M}(\tau)$. \square

LEMMA 4.4. *Let A be a possibly unbounded, not necessarily closed, injective linear operator from $\mathcal{H}(\sigma)$ to $\mathcal{H}(\tau)$ which satisfies all other properties of Definition 4.1. Suppose that $\phi_\delta \in \mathcal{D}(A)$ for all $\delta \in \mathcal{M}(\sigma)$, $\mathcal{M}(\sigma) = \mathcal{M}(\tau)$ and,*

for each $\delta \in \mathcal{M}(\sigma)$, there is a complex number c_δ such that $(Av, \psi_\delta) = c_\delta(v, \phi_\delta)$ for all $v \in \mathcal{D}(A)$. Then the closure \bar{A} of A is one-valued and injective, \bar{A} satisfies all properties of Definition 4.1 and

$$(4.3) \quad \mathcal{D}(\bar{A}) = \{v \in \mathcal{H}(\sigma) \mid \sum_{\delta \in \mathcal{M}(\sigma)} |c_\delta(v, \phi_\delta)|^2 < \infty\}.$$

Proof. Let $\{v_n\}$ be a sequence in $\mathcal{D}(A)$ such that $v_n \rightarrow v$ in $\mathcal{H}(\sigma)$ and $Av_n \rightarrow w$ in $\mathcal{H}(\tau)$. Then, for each $\delta \in \mathcal{M}(\sigma)$,

$$(w, \psi_\delta) = \lim_{n \rightarrow \infty} (Av_n, \psi_\delta) = c_\delta \lim_{n \rightarrow \infty} (v_n, \phi_\delta) = c_\delta(v, \phi_\delta).$$

Hence $v = 0$ iff $w = 0$, so \bar{A} is one-valued and injective.

To prove the domain invariance and intertwining property of \bar{A} , let

$$v \in \mathcal{D}(\bar{A}), \text{ so } v_n \rightarrow v, Av_n \rightarrow \bar{A}v$$

for some sequence $\{v_n\}$ in $\mathcal{D}(A)$. If $g \in G$ then

$$\sigma(g)v_n \rightarrow \sigma(g)v \text{ and } A\sigma(g)v_n = \tau(g)Av_n \rightarrow \tau(g)\bar{A}v,$$

so $\sigma(g)v \in \mathcal{D}(\bar{A})$ and $\bar{A}\sigma(g)v = \tau(g)\bar{A}v$.

Finally, to prove (4.3), first suppose that $v \in \mathcal{H}(\sigma)$ and

$$\sum_{\delta \in \mathcal{M}(\sigma)} |c_\delta(v, \phi_\delta)|^2 < \infty.$$

Then

$$\begin{aligned} v &= \Sigma(v, \phi_\delta)\phi_\delta, w := \Sigma c_\delta(v, \phi_\delta)\psi_\delta \in \mathcal{H}(\tau) \text{ and } \bar{A}\phi_\delta \\ &= c_\delta\psi_\delta, \text{ so, } w = \bar{A}v \text{ and } v \in \mathcal{D}(\bar{A}). \end{aligned}$$

Conversely, let $v \in \mathcal{D}(\bar{A})$. Then $\bar{A}v = \Sigma(\bar{A}v, \psi_\delta)\psi_\delta = \Sigma c_\delta(v, \phi_\delta)\psi_\delta$ (note $(\bar{A}v, \psi_\delta) = c_\delta(v, \phi_\delta)$ by (4.1)). Hence $\Sigma |c_\delta(v, \phi_\delta)|^2 < \infty$. \square

Next we will prove a criterium for Naimark relatedness of K -multiplicity free representations σ and τ in terms of the canonical matrix elements.

THEOREM 4.5. *Let G be an lcsc. group with compact abelian subgroup K . Let σ and τ be K -multiplicity free representations of G . Let $\{\phi_\delta\}$ and $\{\psi_\delta\}$ be K -bases of $\mathcal{H}(\sigma)$ and $\mathcal{H}(\tau)$, respectively. For each $\delta \in \mathcal{M}(\sigma) \cap \mathcal{M}(\tau)$ let $0 \neq c_\delta \in \mathbf{C}$. Then the following two statements are equivalent:*

- (a) $\sigma \xrightarrow{A} \tau$ and $A\phi_\delta = c_\delta \psi_\delta$, $\delta \in \mathcal{M}(\sigma)$.
 (b) $\mathcal{M}(\sigma) = \mathcal{M}(\tau)$ and, for all $\gamma, \delta \in \mathcal{M}(\sigma)$,

$$(4.4) \quad \tau_{\gamma, \delta} = C_{\gamma, \delta} \sigma_{\gamma, \delta}$$

with $C_{\gamma, \delta} = c_\gamma / c_\delta$.

If, moreover, σ and τ are irreducible then (a) and (b) are also equivalent to :

- (c) For some $\gamma, \delta \in \mathcal{M}(\sigma) \cap \mathcal{M}(\tau)$ (4.4) holds for some nonzero complex $C_{\gamma, \delta}$.

Proof.

(a) \Rightarrow (b): Apply Lemma 4.3. By using (4.1) we have

$$\begin{aligned} c_\gamma(\sigma(g)\phi_\delta, \phi_\gamma) &= (A\sigma(g)\phi_\delta, \psi_\gamma) = (\tau(g)A\phi_\delta, \psi_\gamma) \\ &= c_\delta(\tau(g)\psi_\delta, \psi_\gamma). \end{aligned}$$

(b) \Rightarrow (a): Define A on the domain $\{v \in \mathcal{H}(\sigma) \mid \sum |c_\delta(v, \phi_\delta)|^2 < \infty\}$ by $Av := \sum c_\delta(v, \phi_\delta) \psi_\delta$. Then A is injective with dense domain and range and A satisfies (4.1). We will prove that $\mathcal{D}(A)$ is G -invariant and that A is an intertwining operator. Let $v \in \mathcal{D}(A)$, $g \in G$. Then, by (4.4) and the definition of Av :

$$\begin{aligned} c_\gamma(\sigma(g)v, \phi_\gamma) &= c_\gamma \sum c_\delta(v, \phi_\delta) \sigma_{\gamma, \delta}(g) \\ &= \sum c_\delta(v, \phi_\delta) \tau_{\gamma, \delta}(g) = (\tau(g)Av, \psi_\gamma). \end{aligned}$$

Hence

$$\sum |c_\gamma(\sigma(g)v, \phi_\gamma)|^2 = \|\tau(g)Av\|^2 < \infty.$$

So $\sigma(g)v \in \mathcal{D}(A)$ and $A\sigma(g)v = \tau(g)Av$. Now apply Lemma 4.4.

(c) \Rightarrow (b): (σ, τ irreducible): We will first show that $\mathcal{M}(\sigma) = \mathcal{M}(\tau)$ and, for each $\beta \in \mathcal{M}(\sigma)$, $\tau_{\gamma, \beta} = C_{\gamma, \beta} \sigma_{\gamma, \beta}$ and $\tau_{\beta, \delta} = C_{\beta, \delta} \sigma_{\beta, \delta}$ for some nonzero complex $C_{\gamma, \beta}$ and $C_{\beta, \delta}$. It follows from (4.4) evaluated for $g = g_1 k g_2$ that

$$\begin{aligned} &\sum_{\beta \in \mathcal{M}(\tau)} \beta(k) \tau_{\gamma, \beta}(g_1) \tau_{\beta, \delta}(g_2) \\ &= C_{\delta, \gamma} \sum_{\beta \in \mathcal{M}(\sigma)} \beta(k) \sigma_{\gamma, \beta}(g_1) \sigma_{\beta, \delta}(g_2), \quad g_1, g_2 \in G, k \in K. \end{aligned}$$

Both sides are absolutely and uniformly convergent Fourier series in $k \in K$. Because of Theorem 3.2 and the irreducibility of σ and τ , for each $\beta \in \mathcal{M}(\tau)$

respectively $\beta \in \mathcal{M}(\sigma)$ the Fourier coefficient at the left respectively right hand side does not vanish identically in g_1, g_2 . Hence $\mathcal{M}(\sigma) = \mathcal{M}(\tau)$ and

$$\tau_{\gamma, \beta}(g_1)\tau_{\beta, \delta}(g_2) = C_{\gamma, \beta}\sigma_{\gamma, \beta}(g_1)\sigma_{\beta, \delta}(g_2).$$

This implies

$$\tau_{\gamma, \beta} = C_{\gamma, \beta}\sigma_{\gamma, \beta} \text{ and } \tau_{\beta, \delta} = C_{\beta, \delta}\sigma_{\beta, \delta} \text{ with } C_{\gamma, \beta}C_{\beta, \delta} = C_{\gamma, \delta}.$$

By repeating this argument we prove that $\tau_{\alpha, \beta} = C_{\alpha, \beta}\sigma_{\alpha, \beta}$ for all $\alpha, \beta \in \mathcal{M}(\sigma)$ and that $C_{\alpha, \beta}C_{\beta, \delta} = C_{\alpha, \delta}$, i.e. $C_{\alpha, \beta} = C_{\alpha, \delta}/C_{\beta, \delta}$. \square

COROLLARY 4.6. *Let G be an lcsc. group with compact abelian subgroup K . Then Naimark relatedness is an equivalence relation in the class of K -multiplicity free representations of G .*

4.2. THE CASE $SU(1, 1)$

Consider irreducible subquotient representations of $\pi_{\xi, \lambda}$ as classified in Theorem 3.4. By comparing K -contents it follows that the only possible nontrivial Naimark equivalences are:

$$\pi_{\xi, \lambda} \simeq \pi_{\xi, \mu}(\lambda + \xi, \mu + \xi \notin \mathbf{Z} + \frac{1}{2}, \lambda \neq \mu)$$

and

$$\begin{aligned} \pi_{\xi, \lambda}^+ &\simeq \pi_{\xi, -\lambda}^+, \quad \pi_{\xi, \lambda}^0 \simeq \pi_{\xi, -\lambda}^0, \quad \pi_{\xi, \lambda}^- \simeq \pi_{\xi, -\lambda}^- \\ &(\lambda + \xi \in \mathbf{Z} + \frac{1}{2}, \lambda \neq 0). \end{aligned}$$

Suppose that σ and τ are irreducible subquotient representations of $\pi_{\xi, \lambda}$ and $\pi_{\xi, \mu}$, respectively, and that $\phi_m \in \mathcal{H}(\sigma) \cap \mathcal{H}(\tau)$ for some $m \in \mathbf{Z} + \xi$. It follows from Theorem 4.5 that $\sigma \simeq \tau$ iff $\tau_{\xi, \lambda, m, m} = \pi_{\xi, \mu, m, m}$. This last identity already holds if it is valid for the restrictions to A . In view of (2.29) and (2.30) we have: $\sigma \simeq \tau$ iff

$$(4.5) \quad \phi_{2i\lambda}^{(0, 2m)}(t) = \phi_{2i\mu}^{(0, 2m)}(t), \quad t \in \mathbf{R}.$$

Formula (4.5) holds if $\lambda = \pm \mu$ (cf. (2.26)). Conversely, assume (4.5) and expand both sides of (4.5) as a power series in $-(sh t)^2$ by using (2.23) and (2.20). The coefficients of $-(sh t)^2$ yield the equality

$$(m+1+\lambda)(m+1-\lambda) = (m+1+\mu)(m+1-\mu)$$

Hence $\lambda = \pm \mu$. We have proved: