4.1. Naimark equivalence

3.3.5. Further applications of the irreducibility criterium in Theorem 3.2 can be found in MILLER [32, Lemmas 3.2 and 4.5] for the Euclidean motion group of $\mathbf{R}^2$ and for the harmonic oscillator group, TAKAHASHI [39, §3.4] for the discrete series of $SL(2, \mathbf{R})$ and [41, p. 560, Cor. 2] for the spherical principal series of $F_{4(-20)}$.

3.3.6. The method of this section does not show in an *a priori* way that a K-multiplicity free principal series representation has only finitely many irreducible subquotient representations. Actually, this property holds quite generally, cf. WALLACH [45, Theorem 8.13.3].

4. EQUIVALENCES BETWEEN IRREDUCIBLE SUBQUOTIENT REPRESENTATIONS OF THE PRINCIPAL SERIES

4.1. NAIMARK EQUIVALENCE

In this subsection we derive a criterium (Theorem 4.5) for Naimark equivalence of K-multiplicity free representations. Lemmas 4.3 and 4.4 are preparations for its proof.

Let G be an lcsc. group.

Definition 4.1. Let σ and τ be Hilbert representations of G. The representation σ is called *Naimark related* to τ if there is a closed (possibly) unbounded) injective linear operator A from $\mathscr{H}(\sigma)$ to $\mathscr{H}(\tau)$ with domain $\mathscr{D}(A)$ dense in $\mathscr{H}(\sigma)$ and range $\mathscr{R}(A)$ dense in $\mathscr{H}(\tau)$ such that $\mathscr{D}(A)$ is σ-invariant and $A\sigma(g)v = \tau(G)Av$ for all $v \in \mathscr{D}(A)$, $g \in G$. Then we use the notation $\sigma \underset{A}{\simeq} \tau$ or $\sigma \simeq \tau$.

Naimark relatedness is not necessarily a transitive relation (cf. WARNER [48, p. 242]). However, we will see that it becomes an equivalence relation (called *Naimark equivalence*) when restricted to the class of unitary representations or of K-multiplicity free representations, K abelian.

Two unitary representations σ and τ of G are called *unitarily equivalent* if there is an isometry A from $\mathscr{H}(\sigma)$ onto $\mathscr{H}(\tau)$ such that $A\sigma(g)v = \tau(g)Av$ for all $v \in \mathscr{H}(\sigma)$, $g \in G$. Clearly unitary equivalence is an equivalence relation.

PROPOSITION 4.2. *Two unitary representations of an lcsc. group G are Naimark related if and only if they are unitarily equivalent.*

See WARNER [48, Prop. 4.3.1.4] for the proof.

Let K be a compact abelian subgroup of G. Let σ and τ be K-multiplicity free representations of G. Let $\{\phi_\delta\}$ and $\{\psi_\delta\}$ be K-bases for $\mathscr{H}(\sigma)$ and $\mathscr{H}(\tau)$, respectively.

LEMMA 4.3. *If* $\sigma \overset{A}{\simeq} \tau$ *then* $\mathscr{M}(\sigma) = \mathscr{M}(\tau)$, $\phi_\delta \in \mathscr{D}(A)$ *and* $\psi_\delta \in \mathscr{R}(A)$ $(\delta \in \mathscr{M}(\sigma))$, *and there are nonzero complex numbers* $c_\delta(\delta \in \mathscr{M}(\sigma))$ *such that*

$$(Av, \psi_\delta) = c_\delta(v, \phi_\delta), \quad v \in \mathscr{D}(A). \tag{4.1}$$

In particular

$$A\phi_\delta = c_\delta \psi_\delta. \tag{4.2}$$

Proof. Let $\delta \in \mathscr{M}(\sigma)$. Let $v \in \mathscr{D}(A)$. We have, by the intertwining property of A,

$$\int_K \delta(k^{-1})\sigma(k)v\,dk = (v, \phi_\delta)\phi_\delta,$$

$$\int_K \delta(k^{-1})A\sigma(k)v\,dk = \int_K \delta(k^{-1})\sigma(k)Av\,dk$$

$$= \begin{cases} (Av, \psi_\delta)\psi_\delta & \text{if } \delta \in \mathscr{M}(\tau), \\ 0 & \text{if } \delta \notin \mathscr{M}(\tau). \end{cases}$$

Since A is closed, we conclude that $(v, \phi_\delta)\phi_\delta \in \mathscr{D}(A)$ and

$$A((v, \phi_\delta)\phi_\delta) = \begin{cases} (Av, \psi_\delta)\psi_\delta & \text{if } \delta \in \mathscr{M}(\tau), \\ 0 & \text{if } \delta \notin \mathscr{M}(\tau). \end{cases}$$

Since A is injective with dense domain, the left hand side is nonzero for certain $v \in \mathscr{D}(A)$. Hence $\delta \in \mathscr{M}(\tau)$, $\phi_\delta \in \mathscr{D}(A)$ and (4.2) and (4.1) hold for certain nonzero c_δ. Finally, since A is closed with dense range, $\mathscr{M}(\sigma) = \mathscr{M}(\tau)$. □

LEMMA 4.4. *Let A be a possibly unbounded, not necessarily closed, injective linear operator from $\mathscr{H}(\sigma)$ to $\mathscr{H}(\tau)$ which satisfies all other properties of Definition 4.1. Suppose that* $\phi_\delta \in \mathscr{D}(A)$ *for all* $\delta \in \mathscr{M}(\sigma)$, $\mathscr{M}(\sigma) = \mathscr{M}(\tau)$ *and,* *for each* $\delta \in \mathcal{M}(\sigma)$, *there is a complex number* c_δ *such that* $(Av, \psi_\delta) = c_\delta(v, \phi_\delta)$ *for all* $v \in \mathcal{D}(A)$. *Then the closure* $\bar{A}$ *of* A *is one-valued and injective,* $\bar{A}$ *satisfies all properties of Definition* 4.1 *and*

$$\mathcal{D}(\bar{A}) = \{v \in \mathcal{H}(\sigma) \mid \sum_{\delta \in \mathcal{M}(\sigma)} |c_\delta(v, \phi_\delta)|^2 < \infty\}. \tag{4.3}$$

Proof. Let $\{v_n\}$ be a sequence in $\mathcal{D}(A)$ such that $v_n \to v$ in $\mathcal{H}(\sigma)$ and $Av_n \to w$ in $\mathcal{H}(\tau)$. Then, for each $\delta \in \mathcal{M}(\sigma)$,

$$(w, \psi_\delta) = \lim_{n \to \infty} (Av_n, \psi_\delta) = c_\delta \lim_{n \to \infty} (v_n, \phi_\delta) = c_\delta(v, \phi_\delta).$$

Hence $v = 0$ iff $w = 0$, so $\bar{A}$ is one-valued and injective.

To prove the domain invariance and intertwining property of $\bar{A}$, let

$$v \in \mathcal{D}(\bar{A}), \text{ so } v_n \to v,\ Av_n \to \bar{A}v$$

for some sequence $\{v_n\}$ in $\mathcal{D}(A)$. If $g \in G$ then

$$\sigma(g)v_n \to \sigma(g)v \text{ and } A\sigma(g)v_n = \tau(g)Av_n \to \tau(g)\bar{A}v,$$

so $\sigma(g)v \in \mathcal{D}(\bar{A})$ and $\bar{A}\sigma(g)v = \tau(g)\bar{A}v$.

Finally, to prove (4.3), first suppose that $v \in \mathcal{H}(\sigma)$ and

$$\Sigma_{\delta \in \mathcal{M}(\sigma)} |c_\delta(v, \phi_\delta)|^2 < \infty.$$

Then

$$v = \Sigma(v, \phi_\delta)\phi_\delta,\ w := \Sigma c_\delta(v, \phi_\delta)\psi_\delta \in \mathcal{H}(\tau) \text{ and } \bar{A}\phi_\delta$$
$$= c_\delta\psi_\delta, \text{ so, } w = \bar{A}v \text{ and } v \in \mathcal{D}(\bar{A}).$$

Conversely, let $v \in \mathcal{D}(\bar{A})$. Then $\bar{A}v = \Sigma(\bar{A}v, \psi_\delta)\psi_\delta = \Sigma c_\delta(v, \phi_\delta)\psi_\delta$ (note $(\bar{A}v, \psi_\delta) = c_\delta(v, \phi_\delta)$ by (4.1)). Hence $\Sigma |c_\delta(v, \phi_\delta)|^2 < \infty$. □

Next we will prove a criterium for Naimark relatedness of K-multiplicity free representations σ and τ in terms of the canonical matrix elements.

THEOREM 4.5. *Let* G *be an lcsc. group with compact abelian subgroup* K. *Let* σ *and* τ *be K-multiplicity free representations of* G. *Let* $\{\phi_\delta\}$ *and* $\{\psi_\delta\}$ *be K-bases of* $\mathcal{H}(\sigma)$ *and* $\mathcal{H}(\tau)$, *respectively. For each* $\delta \in \mathcal{M}(\sigma) \cap \mathcal{M}(\tau)$ *let* $0 \neq c_\delta \in \mathbf{C}$. *Then the following two statements are equivalent:*

(a) $\sigma \overset{A}{\simeq} \tau$ *and* $A\phi_\delta = c_\delta\psi_\delta$, $\delta \in \mathscr{M}(\sigma)$.

(b) $\mathscr{M}(\sigma) = \mathscr{M}(\tau)$ *and, for all* $\gamma, \delta \in \mathscr{M}(\sigma)$,

$$\tau_{\gamma,\delta} = C_{\gamma,\delta}\sigma_{\gamma,\delta} \tag{4.4}$$

with $C_{\gamma,\delta} = c_\gamma/c_\delta$.

If, moreover, σ *and* τ *are irreducible then* (a) *and* (b) *are also equivalent to:*

(c) *For some* $\gamma, \delta \in \mathscr{M}(\sigma) \cap \mathscr{M}(\tau)$ (4.4) *holds for some nonzero complex* $C_{\gamma,\delta}$.

Proof.

(a) $\Rightarrow$ (b): Apply Lemma 4.3. By using (4.1) we have

$$\begin{aligned} c_\gamma(\sigma(g)\phi_\delta, \phi_\gamma) &= (A\sigma(g)\phi_\delta, \psi_\gamma) = (\tau(g)A\phi_\delta, \psi_\gamma) \\ &= c_\delta(\tau(g)\psi_\delta, \psi_\gamma)\,. \end{aligned}$$

(b) $\Rightarrow$ (a): Define A on the domain $\{v \in \mathscr{H}(\sigma) \mid \Sigma \mid c_\delta(v, \phi_\delta) \mid^2 < \infty\}$ by $Av := \Sigma c_\delta(v, \phi_\delta)\psi_\delta$. Then A is injective with dense domain and range and A satisfies (4.1). We will prove that $\mathscr{D}(A)$ is G-invariant and that A is an intertwining operator. Let $v \in \mathscr{D}(A)$, $g \in G$. Then, by (4.4) and the definition of Av:

$$\begin{aligned} c_\gamma(\sigma(g)v, \phi_\gamma) &= c_\gamma\Sigma_\delta(v, \phi_\delta)\sigma_{\gamma,\delta}(g) \\ &= \Sigma_\delta c_\delta(v, \phi_\delta)\tau_{\gamma,\delta}(g) = (\tau(g)Av, \psi_\gamma)\,. \end{aligned}$$

Hence

$$\Sigma_\gamma \mid c_\gamma(\sigma(g)v, \phi_\gamma) \mid^2 = \| \tau(g)Av \|^2 < \infty\,.$$

So $\sigma(g)v \in \mathscr{D}(A)$ and $A\sigma(g)v = \tau(g)Av$. Now apply Lemma 4.4.

(c) $\Rightarrow$ (b): (σ, τ irreducible): We will first show that $\mathscr{M}(\sigma) = \mathscr{M}(\tau)$ and, for each $\beta \in \mathscr{M}(\sigma)$, $\tau_{\gamma,\beta} = C_{\gamma,\beta}\sigma_{\gamma,\beta}$ and $\tau_{\beta,\delta} = C_{\beta,\delta}\sigma_{\beta,\delta}$ for some nonzero complex $C_{\gamma,\beta}$ and $C_{\beta,\delta}$. It follows from (4.4) evaluated for $g = g_1kg_2$ that

$$\begin{aligned} &\sum_{\beta \in \mathscr{M}(\tau)} \beta(k)\tau_{\gamma,\beta}(g_1)\tau_{\beta,\delta}(g_2) \\ &= C_{\delta,\gamma} \sum_{\beta \in \mathscr{M}(\sigma)} \beta(k)\sigma_{\gamma,\beta}(g_1)\sigma_{\beta,\delta}(g_2)\,, \quad g_1, g_2 \in G, k \in K\,. \end{aligned}$$

Both sides are absolutely and uniformly convergent Fourier series in $k \in K$. Because of Theorem 3.2 and the irreducibility of σ and τ, for each $\beta \in \mathscr{M}(\tau)$

respectively $\beta \in \mathscr{M}(\sigma)$ the Fourier coefficient at the left respectively right hand side does not vanish identically in g_1, g_2. Hence $\mathscr{M}(\sigma) = \mathscr{M}(\tau)$ and

$$\tau_{\gamma,\beta}(g_1)\tau_{\beta,\delta}(g_2) = C_{\gamma,\delta}\sigma_{\gamma,\beta}(g_1)\sigma_{\beta,\delta}(g_2)\,.$$

This implies

$$\tau_{\gamma,\beta} = C_{\gamma,\beta}\sigma_{\gamma,\beta} \text{ and } \tau_{\beta,\delta} = C_{\beta,\delta}\sigma_{\beta,\delta} \text{ with } C_{\gamma,\beta}C_{\beta,\delta} = C_{\gamma,\delta}\,.$$

By repeating this argument we prove that $\tau_{\alpha,\beta} = C_{\alpha,\beta}\sigma_{\alpha,\beta}$ for all $\alpha, \beta \in \mathscr{M}(\sigma)$ and that $C_{\alpha,\beta}C_{\beta,\delta} = C_{\alpha,\delta}$, i.e. $C_{\alpha,\beta} = C_{\alpha,\delta}/C_{\beta,\delta}$. □

COROLLARY 4.6. *Let G be an lcsc. group with compact abelian subgroup K. Then Naimark relatedness is an equivalence relation in the class of K-multiplicity free representations of G.*

4.2. THE CASE $SU(1, 1)$

Consider irreducible subquotient representations of $\pi_{\xi,\lambda}$ as classified in Theorem 3.4. By comparing K-contents it follows that the only possible nontrivial Naimark equivalences are:

$$\pi_{\xi,\lambda} \simeq \pi_{\xi,\mu} (\lambda+\xi,\ \mu+\xi \notin \mathbf{Z}+\tfrac{1}{2},\ \lambda \neq \mu)$$

and

$$\pi^+_{\xi,\lambda} \simeq \pi^+_{\xi,-\lambda},\quad \pi^0_{\xi,\lambda} \simeq \pi^0_{\xi,-\lambda},\quad \pi^-_{\xi,\lambda} \simeq \pi^-_{\xi,-\lambda}$$
$$(\lambda+\xi \in \mathbf{Z}+\tfrac{1}{2},\ \lambda \neq 0)\,.$$

Suppose that σ and τ are irreducible subquotient representations of $\pi_{\xi,\lambda}$ and $\pi_{\xi,\mu}$, respectively, and that $\phi_m \in \mathscr{H}(\sigma) \cap \mathscr{H}(\tau)$ for some $m \in \mathbf{Z} + \xi$. It follows from Theorem 4.5 that $\sigma \simeq \tau$ iff $\tau_{\xi,\lambda,m,m} = \pi_{\xi,\mu,m,m}$. This last identity already holds if it is valid for the restrictions to A. In view of (2.29) and (2.30) we have: $\sigma \simeq \tau$ iff

$$\phi^{(0,\,2m)}_{2i\lambda}(t) = \phi^{(0,\,2m)}_{2i\mu}(t)\,,\qquad t \in \mathbf{R}\,. \tag{4.5}$$

Formula (4.5) holds if $\lambda = \pm\mu$ (cf. (2.26)). Conversely, assume (4.5) and expand both sides of (4.5) as a power series in $-(sh\ t)^2$ by using (2.23) and (2.20). The coefficients of $-(sh\ t)^2$ yield the equality

$$(m+1+\lambda)(m+1-\lambda) = (m+1+\mu)(m+1-\mu)$$

Hence $\lambda = \pm\mu$. We have proved: