5.3. The generalized Abel transform

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **28 (1982)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **25.05.2024**

These are commutative topological algebras under convolution and their characters are precisely of the form (5.1), where ϕ is a spherical function on $G \times K$. If ϕ is a spherical function on $G \times K$ then there is a $\delta \in \hat{K}$ such that for all $x \in G$ the function $k \to \phi(xk)$ on K belongs to δ. Then δ is called a *spherical function of type* δ on G (with respect to K), cf. GODEMENT [19]. It is funny that spherical functions of type δ are on the one hand generalizations of ordinary spherical functions for (G, K), on the other hand restrictions to G of ordinary spherical functions for $(G \times K, K^*)$.

For convenience, we take a one-dimensional $\delta \in \hat{K}$. Then a spherical function ϕ on $G \times K$ is of type δ iff

$$\phi(xk) = \phi(kx) = \delta(k)\phi(x), \qquad x \in G, k \in K .$$

Let

$$\begin{aligned} &I_{c,\delta}(G) \ \left(\text{or } I^{\infty}_{c,\delta}(G)\right) \\ := &\left\{f \in C_c(G) \ \left(\text{or } C_c^{\infty}(G)\right) \mid f(xk) = f(kx) \right. \\ &\left. = \delta(k) f(x),\ x \in G,\ k \in K\right\} . \end{aligned}$$

These are closed subalgebras of $I_c(G)$ (or $I_c^{\infty}(G)$) and their characters are precisely of the form (5.1), where ϕ is a spherical function of type δ. Finally, if τ is a K-unitary representation of G and if $\mathscr{H}(\tau)$ contains a unit vector v satisfying $\tau(k)v = \delta(k)v$, unique up to a constant factor, then $x \to (\tau(x)v, v)$ is a spherical function of type δ.

5.3. The generalized Abel transform

Let G be a connected noncompact real semisimple Lie group with finite center. Use the notation of §2.2. For given Haar measures dk, da, dn on K, A, N, respectively, normalize the Haar measure on G such that

$$\int_G f(g)dg = \int_{K \times A \times N} f(kan) e^{2\rho(\log a)}\, dk\, da\, dn, \quad f \in C_c(G) \tag{5.2}$$

(cf. HELGASON [25, Ch. X, Prop. 1.11]). Note the property

$$\int_N f(n)dn = e^{2\rho(\log a)} \int_N f(ana^{-1})dn, \quad f \in C_c(N),\ a \in A \tag{5.3}$$

(cf. [25, Ch. X, proof of Prop. 1.11]).

For $\lambda \in \mathfrak{a}_{\mathbf{C}}^*$ let U^λ be the representation of G induced by the one-dimensional representation $an \to e^{\lambda(\log a)}$ of the subgroup AN:

(5.4) $\quad (U^\lambda(g)f)(k) := e^{-(\rho+\lambda)H(g^{-1}k)}\, f(u(g^{-1}k)),\ f \in L^2(K),\ g \in G,\ k \in K\,.$

The representation U^λ is easily seen to split as a direct sum of principal series representations $\pi_{\xi,\lambda}$. U^λ restricted to K is the left regular representation of K.

Let $\delta \in \hat{K}$. For convenience, suppose that δ is one-dimensional. The *generalized Abel transform* $f \to F_f^\delta : I_{c,\delta}(G) \to C_c(A)$ is defined by

$$F_f^\delta(a) := e^{\rho(\log a)} \int_N f(an)dn,\ a \in A\,. \tag{5.5}$$

If $G = SU(1, 1)$ and $\delta = 1$ then this transform can be rewritten as the classical Abel transform, cf. §5.4.

PROPOSITION 5.3. *The mapping* $f \to F_f^\delta$ *is a continuous homomorphism (with respect to convolution on* G *and* A, *respectively) from* $I_{c,\delta}^\infty(G)$ *to* $C_c^\infty(A)$. *Furthermore,*

$$\int_A F_f^\delta(a)e^{-\lambda(\log a)}\, da = \int_G f(g)(U^\lambda(g^{-1})\breve{\delta}, \breve{\delta})dg,\ f \in I_{c,\delta}^\infty(G),\ \lambda \in \mathfrak{a}_{\mathbf{C}}^*\,, \tag{5.6}$$

where $(.,.)$ *denotes the inner product on* $L^2(K)$.

Proof. The continuity is immediate. The homomorphism property follows easily from (5.2) and (5.3) (cf. WARNER [49, pp. 34, 35]). For the proof of (5.6) substitute (5.4) into the right hand side of (5.6):

$$\begin{aligned}
\int_G f(g)(U^\lambda(g^{-1})\breve{\delta}, \breve{\delta})dg &= \int_G\int_K f(g)e^{-(\rho+\lambda)H(gk)}\, \breve{\delta}((u(gk))^{-1}k)dk\, dg \\
&= \int_G f(g)e^{-(\rho+\lambda)H(g)}\, \breve{\delta}((u(g))^{-1})dg \\
&= \int_{K\times A\times N} f(kan)e^{(\rho-\lambda)\log a}\, \breve{\delta}(k^{-1})dk\, da\, dn \\
&= \int_A\int_N f(an)e^{(\rho-\lambda)\log a}\, dn\, da \\
&= \int_A F_f^\delta(a)e^{-\lambda(\log a)}\, da\,. \qquad \square
\end{aligned}$$

Now let $G = SU(1, 1)$. Write $F_f^n(t)$ and $I_{c,n}^{\infty}(G)$ instead of $F_f^{\delta_n}(a_t)$ and $I_{c,\delta_n}^{\infty}(G)$, respectively. If $n \in \mathbf{Z} + \xi$ then (5.5) and (5.6) take the form

$$F_f^n(t) = e^{\frac{1}{2}t} \int_{-\infty}^{\infty} f(a_t n_z) dz \tag{5.7}$$

and

$$\int_{-\infty}^{\infty} F_f^n(t) e^{-\lambda t}\, dt = \int_G f(g) \pi_{\xi, \lambda, n, n}(g^{-1}) dg, \; f \in I_{c,n}^{\infty}(G), \lambda \in \mathbf{C}\,, \tag{5.8}$$

where $dg = (2\pi)^{-1} e^t d\theta dt dz$ if $g = u_\theta a_t n_z$.

5.4. The main theorem

It is the purpose of this section to prove:

Theorem 5.4. *Let τ be an irreducible K-unitary representation of $SU(1, 1)$ which is K-finite or unitary. Then τ is Naimark equivalent to an irreducible subrepresentation of some principal series representation $\pi_{\xi,\lambda}$.*

By Proposition 5.2 τ is K-multiplicity free. If $\delta_n \in \mathscr{M}(\tau)$ then write $\tau_{n,n}$ instead of $\tau_{\delta_n, \delta_n}$. In view of Theorem 4.5 and Remark 4.8 it is sufficient for the proof of Theorem 5.4 to show that for some $\delta_n \in \mathscr{M}(\tau)$, for some $\lambda \in \mathbf{C}$ and for $\xi \in \{0, \frac{1}{2}\}$ with $n \in \mathbf{Z} + \xi$ we have

$$\tau_{n,n} = \pi_{\xi, \lambda, n, n}\,. \tag{5.9}$$

Both sides of (5.9) are spherical functions of type δ_n. Then (5.9) holds if the corresponding characters on $I_{c,n}^{\infty}(G)$ are equal. Hence Theorem 5.4 will follow from

Proposition 5.5. *Let $G = SU(1, 1)$, $n \in \frac{1}{2}\mathbf{Z}$. Let α be a continuous character on $I_{c,n}^{\infty}(G)$. Then*

$$\alpha(f) = \int_G f(g) \pi_{\xi, \lambda, n, n}(g^{-1}) dg, \; f \in I_{c,n}^{\infty}(G)\,, \tag{5.10}$$

for some $\lambda \in \mathbf{C}$ and for $\xi \in \{0, \frac{1}{2}\}$ such that $n \in \mathbf{Z} + \xi$.