III. Peano arithmetic and the Stability Condition

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Choose $k > i$ lying in t. Then $h_i < h_k$, so that

$$g_{r(k)}(k) = \beta(r(k), g(k)) < h_k(k) = a_{kn_k}.$$

But by (1) the sequence

$$g_1(k), \ldots, g_{r(k)}(k)$$

satisfies the above formula. This contradicts the minimality of a_{kn_k}.

III. Peano arithmetic and the Stability Condition

Theorem 1 suffices to construct a non-standard model of a theory of arithmetic in which all the axioms are expressed by limited formulas. The induction axioms of Peano arithmetic however involve arbitrary elementary formulas. To deal with this problem we shall associate with each formula $\phi(y)$ of arithmetic a limited formula $\hat{\phi}(y; z)$ [1] called the *limited associate* of $\phi(y)$.

We assume that $\phi(y)$ has been reduced to prenex normal form. To obtain the formula $\hat{\phi}(y; z)$ we replace each quantifier Qx_i in $\phi(y)$ by the bounded quantifier $Qx_i < z_i$. The bounding variables z_k are to be distinct from the variables occurring in $\phi(y)$ and also distinct from each other.

Although Theorem 1 allows us to prove the validity of limited associates of the Peano axioms in the model $\mathscr{F}/D$, we need a provision for inferring from this the validity of the Peano axioms themselves in $\mathscr{F}/D$.

To obtain the desired result it would suffice to show that for some suitable vector h in $\mathscr{F}$, $\mathscr{F}/D \models \hat{\phi}(y; h^*)$ implies $\mathscr{F}/D \models \phi(y)$. However, if we consider the case where $\phi(y)$ is $(\forall x)(y \neq x)$, we find $\mathscr{F}/D \models \hat{\phi}(h^*, h^*)$ but $\mathscr{F}/D \models \neg \phi(h^*)$. This example shows that we must restrict the range of y, i.e. require that for all $f < h$, $\mathscr{F}/D \models \hat{\phi}(f^*, h^*)$ implies $\mathscr{F}/D \models \phi(f^*)$. To prove $\mathscr{F}/D \models \phi(f^*)$ for all f in $\mathscr{F}$ it thus suffices to construct an increasing sequence $\{h_i\}$ in $\mathscr{F}$, cofinal in $\mathscr{F}$, such that for all i, j with $i < j$ [2] and all $f \in \mathscr{F}$ with $f < h_i$,

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[1]) Here and later y and z denote vectors of variables.

[2]) Here and later j denotes a vector $\langle j_1, \ldots, j_n \rangle$, $i < j$ means $i < \min j$, and h_j denotes the vector $\langle h_{j_1}, \ldots, h_{j_n} \rangle$.

$$\mathscr{F}/D \models \hat{\phi}(f^*; h_j^*) \text{ if and only if } \mathscr{F}/D \models \phi(f^*). \tag{1}$$

Now the equivalence (1) refers to a formula $\phi(y)$ with unbounded quantifiers and so is not a tractable condition to handle via Theorems 1 and 2. We shall accordingly replace (1) by an equivalent condition which refers only to limited formulas. To see what this condition is consider the case where $\phi(y)$ is a formula of the form $(\forall x)\,\psi(x, y)$, where ψ is quantifier free. Suppose that for some i, j with $i < j$ and all $f \in \mathscr{F}$ with $f < h_i$ we have

$$\mathscr{F}/D \models \hat{\phi}(f^*; h_j^*)$$
$$\text{i.e.} \quad \mathscr{F}/D \models (\forall x < h_j^*)\,\psi(f^*, x)$$

If condition (1) holds then

$$\mathscr{F}/D \models (\forall x)\,\psi(f^*, x)$$

and hence

$$\mathscr{F}/D \models (\forall x < h_{j'}^*)\,\psi(f^*, x)$$

for all $j' > i$.

Thus, for condition (1) to hold it is necessary for the truth value of the limited formula $\hat{\phi}(y; z)$ to eventually stabilize. We formulate this condition as follows.

We shall assume for the rest of the paper that the sequence $h_j = \langle h_{j1}, \ldots, h_{jn} \rangle$ substituted for the bounding variables $z = \langle z_{i_1}, \ldots, z_{i_n} \rangle$ in the limited formula $\psi(y; z)$ is an increasing sequence (i.e. $r < s$ implies $j_r < j_s$ and hence $h_{j_r} < h_{j_s}$). Thus, the smaller the scope of the bounded quantifier $(Qx_{i_r} < z_{i_r})$ in $\psi(y; z)$ the larger the substituted element $h_{j_r}^*$ in the sequence $h_j^* = \langle h_{j1}^*, \ldots, h_{jn}^* \rangle$.

Stability Condition. For every limited formula $\psi(y; z)$ and for all i, j, j' with $i < j$, $i < j'$

$$\mathscr{F}/D \models (\forall y < h_i)\,(\psi(y; h_j^*) \leftrightarrow \psi(y; h_{j'}^*)).$$

We shall now prove that this condition suffices to establish (1) (c.f. [3] Proposition 2.2).

LEMMA 1. *Assume that $\{h_i\}$ is an increasing cofinal sequence in $\mathscr{F}$ which satisfies the Stability Condition. Then for all $i < j$*

$$\mathscr{F}/D \models (\forall y < h_i^*)\,(\phi(y; h_j^*) \Leftrightarrow \hat{\phi}(y; h_j^*)).$$

Proof: We shall assume that $\phi(y)$ is already in prenex normal form. We proceed by induction on the number of quantifiers occurring in $\phi(y)$. For quantifier-free formulas the equivalence is clearly true.

Now assume that $\phi(y)$ has the form $(\exists x_1)\,\psi(x_1, y)$. Then $\hat{\phi}(y; z)$ has the form $(\exists x_1 < z_1)\,\hat{\psi}(x_1, y; z_2, \dots, z_e)$. $\mathscr{F}/D \models \phi(f^*)$ if and only if $\mathscr{F}/D \models \hat{\psi}(b^*, f^*)$ for some $b \in \mathscr{F}$.

By induction it follows that $\mathscr{F}/D \models \phi(f^*)$ if and only if $\mathscr{F}/D \models \hat{\psi}(b^*, f^*, h^*_{j_2}, \dots, h^*_{j_e})$ for all $j_2, \dots, j_e$, with $i < j_2 < \dots < j_e$. By cofinality $b < h_{j_1}$, for some $j_1 > i$. Hence

$$\mathscr{F}/D \models \phi(f^*) \text{ if and only if } \mathscr{F}/D \models (\exists x_1 < h^*_{j_1})\,\hat{\psi}(x_1, f^*; h^*_{j_2} \dots h^*_{j_e})$$
$$\text{i.e. } \mathscr{F}/D \models \hat{\phi}(f^*; h^*_{j_0}), \text{ where } j_o = \langle j_1, j_2, \dots, j_e \rangle .$$

By the Stability Condition, $\mathscr{F}/D \models \phi(f^*; h^*_{j_0})$ for this $j_o > i$ is equivalent to $\mathscr{F}/D \models \hat{\phi}(f^*; h^*_j)$ for all $j > i$, completing the induction step.

THEOREM 4. *Assume that* $\{h_i\}$ *is a cofinal sequence in* $\mathscr{F}$ *with* $h_i^2 < h_{i+1}$ *which obeys the Stability Condition. Then* $\mathscr{F}/D$ *is a model of the Peano axioms.*

Proof. The axioms $\forall x\, \forall y\, \exists z\, \sigma(x, y, z)$ and $\forall x\, \forall y\, \exists z\, \pi(x, y, z)$ are valid because $\mathscr{F}$ is closed under $+$ and $\cdot$.

Every other non-induction Peano axiom ϕ is a $\prod^0_1$-statement. Thus $\mathbf{N} \models \hat{\phi}(z)$. By Theorem 1, $\mathscr{F}/D \models \hat{\phi}(z)$. Hence $\mathscr{F}/D \models \phi$.

Now let $\phi(y)$ be the induction formula

$$[\psi(0, y) \wedge (\forall x)(\psi(x, y) \rightarrow \psi(x+1, y))] \rightarrow (\forall x)\,\psi(x, y) \ .$$

We may assume that $\psi(x, y)$ is in prenex normal form.

Note that for any formula $\eta(x)$

$$\mathbf{N} \models [\eta(0) \wedge (\forall x < w)(\eta(x) \rightarrow \eta(x+1))] \rightarrow (\forall x < w)\,\eta(x) \, .$$

Hence, if $\eta(x)$ is a limited formula then Theorem 1 implies that

$$\mathscr{F}/D \models [\eta(0) \wedge (x < w)\,(\eta(x) \rightarrow \eta(x+1))] \rightarrow (\forall x < w)\,\eta(x) \ .$$

In particular, taking for η the limited associate $\hat{\psi}(x, y; z)$ of $\psi(x, y)$, we have that

$$F/D \models [\hat{\psi}(0, y; z) \wedge (\forall x < w)\big(\hat{\psi}(x, y; z) \rightarrow \hat{\psi}(x+1, y; z)\big)] \\ \rightarrow (\forall x < w)\,\hat{\psi}(x, y; z) \tag{1}$$

We now assume that $\mathscr{F}/D \models \psi(0, g^*)$ and $\mathscr{F}/D \models (\forall x)\big(\psi(x, g^*)) \rightarrow \psi(x+1, g^*)\big)$ for some vector g of functions in $\mathscr{F}$.

We have $g < h_i$ (i.e. $\max g < h_i$) for some i. Choose any j, t with $j > t > i$. By Lemma 1, $\mathscr{F}/D \models \psi(0, g^*; h_j^*)$. Assume for $x < h_t^*$ that

$$\mathscr{F}/D \models \hat{\psi}(x, g^*; h_j^*)\,.$$

By Lemma 1,

$$\mathscr{F}/D \models \psi(x, g^*)\,.$$

Hence,

$$\mathscr{F}/D \models \psi(x+1, g^*)$$

so that, again by Lemma 1,

$$\mathscr{F}/D \models \hat{\psi}(x+1, g^*; h_j^*)\,.$$

Thus, by (1),

$$\mathscr{F}/D \models (\forall x < h_t^*)\,\hat{\psi}(x, y^*; h_j^*)\,.$$

It follows from Lemma 1 that

$$\mathscr{F}/D \models \forall x\, \psi(x, g^*)\,.$$

We have thus proved that the induction formula $\phi(y)$ is valid in $\mathscr{F}/D$.

It remains for us to construct a suitable sequence $\{h_i\}$ of functions satisfying the Stability Condition. Let $\{\psi_j\}$ be an effective enumeration of all the limited formulas. The Stability and Closure Conditions have the form

$$\mathscr{F}/D \models \exists z_1 \ldots \exists z_n \ldots \bigwedge_{\substack{1 \leq i < j,\, j' < \infty \\ 1 \leq s < \infty}} [(\forall z < z_i)\big(\psi_s(y; z_j) \\ \leftrightarrow \psi_s(y; z_{j'})\big) \wedge z_{j-1}^2 < z_j]$$

Now this condition has precisely the form needed for the conclusion of the Saturation Theorem 2. Thus, if we can show that for each k

$$\mathbf{N} \models \exists z_1 \ldots \exists z_{n_k} \bigwedge_{\substack{1 \leq i < j,\, j' < n_k \\ 1 \leq s < k}} [(\forall y < z_i)\big(\psi_s(y; z_j) \\ \leftrightarrow \psi_s(y; z_{j'})\big) \wedge z_{j-1}^2 < z_j] \tag{*}$$

then we can construct the sequence $\{ h_i \}$ and the set $\mathscr{F}$ to satisfy the Stability and Closure Conditions via Theorem 2.

We could now proceed to show that the above condition is indeed satisfied in $\mathbf{N}$ and thus construct a non-standard model of Peano arithmetic. However, our goal is the construction of a mathematically perspicuous model which is independent of the logical formulas. The functions $\{ h_i \}$ given by the above condition require the logical calculus in their definition. Accordingly, we shall consider a larger class $\mathscr{F}$ of functions than those defined above, which we shall construct independently of logical formulas. This class will be constructed from combinatorial principles derived from Ramsey's Partition Theorem.

IV. Ramsey-type Theorems

The infinite Ramsey Theorem states that for every partition $P : [\mathbf{N}]^e \to r$ [1]) there exists an infinite subset X of $\mathbf{N}$ such that $P \mid [X]^e$ is constant. In these circumstances one says that X is homogeneous for the partition P. This set-theoretic theorem has various combinatorial consequences which are formalizable in elementary arithmetic. One such immediate consequence which we shall prove independent of the Peano axioms is the following.

Proposition 1. *Let* $P : [\mathbf{N}]^e \to r$ *be a primitive recursive partition. For every natural number* k *there exists a finite subset* X *of* $\mathbf{N}$, *with* $\# X \geqslant k$ *and* $\# X \geqslant 2^{2^{\min X}}$, *which is homogeneous for the partition* P.

In order to apply Theorem 2 we require the construction of a set which is simultaneously homogeneous for several partitions. This is easily done by the infinite Ramsey Theorem. Suppose $P_1 : [\mathbf{N}]^{e_1} \to r_1$ and $P_2 : [\mathbf{N}]^{e_2} \to r_2$ are two partitions. Let X_1 be an infinite subset of $\mathbf{N}$ homogeneous for P_1. Then $P_2 \mid [X]^{e_2}$ is a partition of $[X_1]^{e_2}$, and hence there is is an infinite subset X_2 of X_1 which is homogeneous for P_2 (as well as P_1). This proof extends immediately to finitely many partitions. A direct consequence is the following generalization of Proposition 1.

Proposition 2. *Let* $P_i : [\mathbf{N}]^{e_i} \to r_i$, $i \leqslant i \leqslant n$ *be a set of primitive recursive partitions. For every natural number* k *there exists a finite subset* X *of* $\mathbf{N}$ *with* $\# X \geqslant k$ *and* $\# X \geqslant 2^{2^{\min X}}$, *which is simultaneously homogeneous for all the partitions* $P_1, \ldots, P_n$.

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[1]) We identify the number r with the set of all natural numbers $< r$.