

# VI. A Simpler Model

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We have thereby shown that  $\mathcal{F}/D$  is a model of the Peano axioms. Since  $a_{kn_k}$  was chosen minimal, Proposition 2 is false in  $\mathcal{F}/D$ , and hence independent of the Peano axioms.

Proposition 1 is also false in  $\mathcal{F}/D$ . In fact it is provable in Peano arithmetic that Proposition 1 implies Proposition 2. This is a consequence of the following lemma, provable in Peano arithmetic (c.f. Lemma 2.9 in [3]).

LEMMA 2. *Let  $P_i : [\mathbb{N}]^e \rightarrow r_i$ ,  $1 \leq i \leq n$ , be  $n$  partitions. There is a partition  $P : [\mathbb{N}]^e \rightarrow r$  such that for all subsets  $H$  of  $\mathbb{N}$  of cardinality  $> e$ ,  $H$  is homogeneous for  $P$  if and only if  $H$  is homogeneous for all the  $P_i$ .*

We may also obtain a purely finitary combinatorial principle which is false in our model.

PROPOSITION 3. *For all natural numbers  $e$ ,  $r$ , and  $k$  there exists an  $N$ , such that for all partitions  $P : [\mathbb{N}]^e \rightarrow r$  there exists a subset  $X$  of  $N$ , with  $\#X \geq k$  and  $\#X \geq 2^{2^{\min X}}$ , which is homogeneous for  $P$ .*

This result follows immediately from the infinite Ramsey Theorem by an application of König's Lemma. If we drop the condition that  $\#X \geq 2^{2^{\min X}}$ , then we obtain the usual finite Ramsey Theorem. Ramsey [11] gave a proof of the latter theorem which is formalizable in Peano arithmetic. Proposition 3 directly yields Proposition 1, for if  $P : [\mathbb{N}]^e \rightarrow r$  is a partition and  $k$  is a number then by considering the partition  $P \upharpoonright [N]^e$ , where  $N$  is the number provided by Proposition 3 we obtain the required homogeneous set  $X$  for  $P \upharpoonright [N]^e$  and hence for  $P$ . This proof may be carried out in Peano arithmetic. Thus, Proposition 3 is false in our model and independent of the Peano axioms.

## VI. A SIMPLER MODEL

The condition in Proposition 1 that  $\#X \geq 2^{2^{\min X}}$  can be simplified and so yield a simpler sequence  $\{h_i\}$  of functions which define the model  $\mathcal{F}/D$ . In this section we describe such a model by using a combinatorial consequence of Ramsey's Theorem which is closer to the proposition proved independent in [3].

PROPOSITION 4. *Let  $P : [\mathbb{N}]^e \rightarrow r$  be a primitive recursive partition. For every  $k$  there exists a finite subset  $X$  of  $\mathbb{N}$ , with  $\#X \geq k$  and  $\#X \geq \min X$ , which is homogeneous for the partition  $P$ .*

Proposition 4 implies Proposition 1 via the following result, the proof of which is the same as the proof of Lemma 2.14 of [3].

LEMMA 3. Let  $P : [\mathbb{N}]^e \rightarrow r$  ( $e \geq 2$ ) be a partition. There is a partition  $P^* : [\mathbb{N}]^e \rightarrow r^*$  (where  $r^*$  depends only on  $m$ ,  $e$ , and  $r$ ) such that if  $X^*$  is a finite subset of  $\mathbb{N}$ , homogeneous for  $P^*$  with  $\# X^* \geq e+1$  and  $\# X^* \geq \min X$ , then the set  $X = [\log_2 \log_2](X^*)$  is homogeneous for  $P$ , and

$$\# X \geq e + 1 \quad \text{and} \quad \# X \geq 2^{2^{\min X}}.$$

Moreover, if  $P$  is a primitive recursive partition, then  $P^*$  can be chosen to be primitive recursive.<sup>1)</sup>

Since this proof that Proposition 4 implies Proposition 1 may be carried out in Peano arithmetic, it follows that Proposition 4 is also false in our model  $\mathcal{F}/D$ . However, our aim here is not merely to give a simple independent statement but to construct a simpler model for Peano arithmetic. Once again we actually use a version of the combinatorial principle which applies to several partitions. The following result is implied by Proposition 4 in Peano arithmetic.

PROPOSITION 5. Let  $P_i : [\mathbb{N}]^{e_i} \rightarrow r_i$ ,  $1 \leq i \leq n$ , be a set of primitive recursive partitions. For every  $k$  there exists a finite subset of  $\mathbb{N}$ , with  $\# X \geq k$  and  $\# X \geq \min X$ , which is simultaneously homogeneous for all the partitions  $P_1, \dots, P_n$ .

We now construct a non-standard model via Proposition 5. Let  $\{P_i\}$  again be an effective enumeration of all the primitive recursive partitions  $P_i : [\mathbb{N}]^{e_i} \rightarrow r_i$ . Let  $c_{k1}, \dots, c_{kn_k}$  be an increasing sequence with  $c_{kn_k}$  the least number such that  $c_{k1}, \dots, c_{kn_k}$  is homogeneous for all  $P_1, \dots, P_k$ , with  $c_{k1} \leq n_k$  and  $k \leq n_k$ . Define the functions  $g_j$  by

$$g_0(k) = n_k \quad \text{for every } k$$

and for  $j > 0$

$$g_j(k) = \begin{cases} c_{kj} & \text{for } j \leq n_k \\ g_{j-1}(k)^2 & \text{for } j > n_k. \end{cases}$$

$$\text{Let } \mathcal{F} = \{f \mid \exists j f \leq g_j\}.$$

We shall show that  $\mathcal{F}/D$  is a model of Peano arithmetic by proving that there is an increasing sequence  $\{h_j\}$  which lies in and is cofinal with  $\mathcal{F}$  and which satisfies the Stability and Closure Conditions. We set

$$h_j = [\log_2 \log_2 g_j].$$

<sup>1)</sup> Here, as is customary,  $[x]$  is the greatest integer  $\leq x$ .

Since  $h_j < g_j$ ,  $h_j \in \mathcal{F}$ . It follows from Lemma 2.13 of [3] that there is a primitive recursive partition  $R$  such that if  $X$  is homogeneous for  $R$ , with  $\# X \geq \min X$  and  $\# X \geq 3$ , then for every  $x, y \in X$ ,  $x < y$  implies  $2^{2^x} < y$ . Since this partition appears in the enumeration  $\{P_i\}$  at some point  $k$ , it follows that, for all  $i \geq k$  and  $j < n_i$ ,  $2^{2^{g_j(i)}} < g_{j+1}(i)$ . Thus, if for a given  $j$  we choose an  $m \geq k$  such that  $n_m \geq j$ , then, for all  $i \geq m$ ,  $2^{2^{g_j(i)}} < g_{j+1}(i)$ . For every  $i < m$  choose an  $s_i$  with  $2^{2^{g_j(i)}} < g_{s_i}(i)$ . Let

$$s = \max(s_1, \dots, s_{m-1}, j+1)$$

Then

$$2^{2^{g_i}} < g_s.$$

Thus  $h_s = [\log_2 \log_2 g_s] > g_j$ , proving that  $\{h_i\}$  is cofinal in  $\mathcal{F}$ .

For each partition  $P_k$  in the sequence  $\{P_i\}$  there exists another partition  $P_t (= P_k^*)$  satisfying the conditions of Lemma 3. By the definition of the functions  $g_j$ , the set  $\{g_1(t), \dots, g_{n_t}(t)\}$  is homogeneous for  $P_t$  and  $n_t \geq t$ ,  $n_t \geq g_1(t)$ . Hence, by Lemma 3, the set

$$\{h_1(t), \dots, h_{n_t}(t)\} = \{[\log_2 \log_2 g_1(t)], \dots, [\log_2 \log_2 g_{n_t}(t)]\}$$

is homogeneous for  $P_k$  and  $n_t \geq 2^{2^{h_1(t)}}$ . Thus, as in the previous section, the sequence  $\{h_j\}$  fulfills the conditions which ensure the satisfaction of the Stability and Closure Conditions. This proves that  $\mathcal{F}/D$  is a model of the Peano axioms. Once again, since  $c_{kn_k}$  was chosen as minimal, it follows that Proposition 5, and hence Proposition 4, is false in  $\mathcal{F}/D$ , and therefore independent of Peano arithmetic.

As before we may formulate a finite version of this combinatorial principle.

**PROPOSITION 6.** *For every  $e, r$ , and  $k$  there exists an  $N$  such that for every partition  $P : [N]^e \rightarrow r$  there exists a subset  $X$  of  $N$ , with  $\# X \geq k$  and  $\# X \geq \min X$ , which is homogeneous for  $P$ .*

Again it is provable in Peano arithmetic that Proposition 6 implies Proposition 4, so that Proposition 6 is false in our model. Proposition 6 was first proved independent of Peano arithmetic in [3] by showing that it implies the consistency of Peano arithmetic and then applying Gödel's Theorem.

Let  $C_k = \{i \mid i \leq c_{kn_k}\}$ . The model  $\mathcal{F}/D$  is an initial segment not only of the ultrapower  $\mathbf{N}^I/D$  but also of the smaller ultraproduct  $\prod_{k \in \mathbf{N}} C_k/D$ .

This indicates that the function  $C$  given by  $C(k) = c_{kn_k}$  is a very rapidly growing function. In fact the function  $C$  majorizes every recursive function which is a provably total function in Peano arithmetic.

**THEOREM 5.** *Let  $f$  be a recursive function. Let  $\psi$  be an elementary statement expressing the condition that  $f$  is a total function. If  $\psi$  is provable in Peano arithmetic,  $f(k) < C(k)$  for all sufficiently large  $k$ .*

*Proof.* Suppose  $t = \{k \mid f(k) \geq C(k)\}$  is infinite. Let  $D$  be a non-principal ultrafilter such that  $t \in D$ . Then  $f^* \geq C^*$ . On the other hand,  $f^* = f(1^*) \in \mathcal{F}/D$ , so that  $f^* < C^*$ , a contradiction.

It follows a fortiori that if  $N$  is the smallest integer to satisfy Theorem 5 then this function  $N$  also majorizes every provably total recursive function (c.f. Theorem 3.2 in [3]).

We mentioned in the introduction that a by-product of our construction is a new proof of Specker's theorem that there exists a recursive partition with no recursively enumerable infinite homogeneous set. In fact we may obtain the stronger theorem that for each  $e \geq 2$ , there exists a primitive recursive partition:  $P : [\mathbb{N}]^e \rightarrow 2$  such that  $P$  has no infinite homogeneous set in  $\sum_e^0$  (c.f. Jockusch [10], Theorem 5.1). We outline the proof of this result. Let  $\phi(y)$  be any formula. As in Section III, the limited associate  $\hat{\phi}(y; z)$  of  $\phi(y)$  defines a partition  $P : [\mathbb{N}]^e \rightarrow 2$  such that every sequence  $\{b_i\}$  of natural numbers homogeneous for  $P$  satisfies the Stability Condition for  $\hat{\phi}(y; z)$  in  $\mathbb{N}$ . Hence, for any vector  $a$  in  $\mathbb{N}$   $\phi(a)$  holds in  $\mathbb{N}$  if and only if  $\hat{\phi}(a; b)$  does. It follows that the set  $\{a \mid \mathbb{N} \models \phi(a)\}$  is recursive in the set  $\{b_i\}$ . Thus the set  $\{b_i\}$  is not in  $\sum_e^0$ .

## VII. VARIATIONS

We conclude with a series of remarks on various modifications of our construction.

(a) It is easily proved that if  $\mathcal{F}$  is closed under  $<$  and contains 1, then  $\mathcal{F}/D$  is non-denumerable, for every non-principal ultrafilter  $D$ . Thus, this construction leads only to non-denumerable models. However, a slight variation of the basic construction yields denumerable models. Note that in the proof of Theorem 1 the function  $g$  is primitive recursive in  $f$ . It follows that we may define  $\mathcal{F} = \{f \mid \exists j f \leq h_j \text{ and } f \text{ is primitive recursive}$