1. The sheaf representation of Boolean algebra extensions

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1. The sheaf representation of Boolean algebra extensions

Let $\mathscr{L}$ be any language for first-order predicate logic. Suppose X is a non-empty set and for every $p \in X$ we have an $\mathscr{L}$-structure $\mathscr{B}_p = (B_p, \ldots)$; put $S = \bigcup_{p \in X} B_p$. Suppose $\varphi(x_1 \ldots x_n)$ is an $\mathscr{L}$-formula, $u \subseteq X$ and $f_1, \ldots, f_n : u \to S$ are such that $f_i(p) \in B_p$ for $1 \leqslant i \leqslant n$ and $p \in u$. Then let

$$\| \varphi [f_1 \ldots f_n] \| = \{ p \in u \mid \mathscr{B}_p \models \varphi [f_1(p) \ldots f_n(p)] \} .$$

We may think of $\| \varphi [f_1 \ldots f_n] \| \subseteq X$ as being a (Boolean) truth value of $\varphi [f_1 \ldots f_n]$ in the power set of X.

A sheaf of $\mathscr{L}$-structures is a sequence

$$\mathscr{S} = (S, \pi, X, \mu)$$

such that a) S and X are topological spaces and $\pi : S \to X$ is a continuous open local homeomorphism from S onto X, b) μ is a function assigning to each $p \in X$ an $\mathscr{L}$-structure $\mathscr{B}_p = (B_p, \ldots)$ such that the B_p are pairwise disjoint, $S = \bigcup_{p \in X} B_p$ and $\pi(s) = p$ iff $s \in B_p$, c) for every open subset u of X and continuous $f_1, \ldots, f_n : u \to S$ satisfying $f_i(p) \in B_p$ for $p \in u$ and every atomic $\mathscr{L}$-formula $\varphi(x_1 \ldots x_n)$, $\| \varphi [f_1 \ldots f_n] \|$ is an open subset of u.

The $\mathscr{L}$-structure $\mathscr{B}_p$ is called the stalk of $\mathscr{S}$ over p. — Let, if $\mathscr{S}$ is a sheaf of $\mathscr{L}$-structures, $\Gamma(\mathscr{S})$ be the set of all continuous functions $f : X \to S$ satisfying $f(p) \in B_p$ for $p \in X$ (the set of "global sections" of $\mathscr{S}$). So $\Gamma(\mathscr{S})$ is, if non-empty, (the underlying set of) a substructure of the product structure $\prod_{p \in X} \mathscr{B}_p$, hence an $\mathscr{L}$-structure.

For the rest of the paper, let $\mathscr{L} = \{ +, \cdot, -, 0, 1, U \}$ where U is a unary predicate. We indicate how, for a given BA extension (B, A), B may be represented by $\Gamma(\mathscr{S})$ where $\mathscr{S}$ is a sheaf of $\mathscr{L}$-structures over a Boolean space. We omit most of the proofs which are easy and entirely analoguous to well-known representation theorems for lattices over Boolean spaces. Let X be the Stone space of A, i.e. the set of all ultrafilters of A with the usual topology. For $p \in X$, let $< p >$ be the filter of B generated by p. Let $\pi_p : B \to B/ < p > = B_p$ be the canonical epimorphism. So B_p is a BA with at least two elements. For $p, q \in X$ and $p \neq q$, B_p and B_q are disjoint. Let $S = \bigcup_{p \in X} B_p$ and $\pi : S \to X$ be defined as stated in b) above. Let, for $p \in X$, $\mu(p)$ be the $\mathscr{L}$-structure $(B_p, \ldots, \{0, 1\})$. For $u \subseteq X$ open and $b \in B$, let $M_{ub} = \{ \pi_p(b) \mid p \in u \}$. The set of these M_{ub} constitutes a base for a topology of S, and this makes $\mathscr{S} = (S, \pi, X, \mu)$ a sheaf of $\mathscr{L}$-structures. Furthermore, for $b \in B$, $\sigma_b : X \to S$ defined by $\sigma_b(p) = \pi_p(b)$ is a global section of $\mathscr{S}$ and

$$\left.\begin{array}{l} \sigma : B \to \Gamma(\mathscr{S}) \\ \quad b \mapsto \sigma_b \end{array}\right\}$$

is an isomorphism from B onto $\Gamma(\mathscr{S})$. We shall now identify B with $\Gamma(\mathscr{S})$; so every $b \in B$ is a function from X to S. This identifies A with those $b \in B$ such that for every $p \in X\, b(p) = 0$ or $b(p) = 1$, i.e. with those $b \in B$ satisfying $\| U(b) \| = X$. Let C be the BA of clopen subsets of X and $e(c)$ the characteristic function of c for $c \in C$. Thus e is an isomorphism from C onto $A \subseteq B$.

In the rest of this section, we show that the property of being a Hausdorff sheaf for $\mathscr{S}$ is equivalent to a property of the extension (B, A) which reflects, in a way which is first-order expressible in $\mathscr{L}$, completeness of the embedding of A into B. Recall that, for a sheaf $\mathscr{S}$ over a Boolean space X, S is a T_2-space iff, for any $f, g \in \Gamma(\mathscr{S})$, $\| f = g \|$ is a clopen subset of X; $\mathscr{S}$ is then said to be a Hausdorff sheaf. Call A relatively complete in B if, for every $b \in B$, there is a largest element $a \in A$ such that $a \leqslant b$, equivalently: for $b \in B$, there is a largest $a \in A$ such that $a \cdot b = 0$ or: for $b \in B$, there is a smallest $a \in A$ such that $b \leqslant a$.

1.1. PROPOSITION. *$\mathscr{S}$ is a Hausdorff sheaf iff A is relatively complete in B.*

Proof. Suppose $\mathscr{S}$ is Hausdorff and $b \in B$. Let $d \in B$ such that $d(p) = 0$ for every $p \in X$, let $c = \| b = d \|$ and $a = e(c)$. Then a is the largest element of A satisfying $a \cdot b = 0$.

Conversely, let A be relatively complete in B and suppose $f, g \in B$. Let a be the largest element of A such that $a \leqslant f \cdot g + -f \cdot -g$. Let $c \in C$ such that $a = e(c)$. Then $\| f = g \| = c$ is a clopen subset of X.

1.2. REMARK. Let A be relatively complete in B. Then the inclusion map from A to B is a complete homomorphism.

Proof. Suppose M is a subset of A having a supremum a in A. We show that a is the supremum of M in B. Clearly, a is an upper bound for M in B. Suppose that b is another upper bound for M in B. Let $\alpha \in A$ be the largest element of A such that $\alpha \leqslant b$. For every $m \in M \subseteq A$, we have $m \leqslant b$, hence $m \leqslant \alpha$ and $a \leqslant \alpha \leqslant b$.

The following facts are trivial:

1.3. REMARK. *a)* Let A and the inclusion map from A to B be complete. Then A is relatively complete in B.

b) Suppose A is relatively complete in B and B is complete. Then A is complete.

2. RELATIVE AUTOMORPHISMS OF FINITE EXTENSIONS

We first give an internal description of a finite extension (B, A) where $B = A(u_1 \dots u_n)$ and $n \in \omega$. We shall always assume that $u_1, \dots, u_n$ are the atoms of the subalgebra of B generated by $u_1, \dots, u_n$; i.e. that they are non-zero, pairwise disjoint and $u_1 + \dots + u_n = 1$. Let $I_r = \{a \in A \mid a \cdot u_r = 0\}$ for $1 \leqslant r \leqslant n$. Clearly, each I_r is a proper ideal of A and $I_1 \cap \dots \cap I_n = \{0\}$. The family $(I_r \mid 1 \leqslant r \leqslant n)$ completely characterizes the extension (B, A):

2.1. REMARK. Suppose $C = A(v_1 \dots v_n)$ is a finite extension of A where $v_1, \dots, v_n$ are pairwise disjoint and $1 = v_1 + \dots + v_n$. Let $B = A(u_1 \dots u_n)$ be as above. There is an isomorphism g from B onto C satisfying $g(a) = a$ for $a \in A$ and $g(u_r) = v_r$ iff, for each r, $\{a \in A \mid a \cdot v_r = 0\} = I_r$.

Proof. By Theorem 12.4 in [7].

2.2. REMARK. A is relatively complete in $B = A(u_1 \dots u_n)$ iff, for each r, I_r is a principal ideal.

Proof. The only—if part follows by the definition of relative completeness. Now suppose $\alpha_r \in A$ generates I_r; let $b \in B$ and $I = \{a \in A \mid a \cdot b = 0\}$. There are $a_1, \dots, a_n \in A$ such that $b = a_1 \cdot u_1 + \dots + a_n \cdot u_n$. It follows that I is the principal ideal generated by $\alpha = (-a_1 + \alpha_1) \cdot \dots \cdot (-a_n + \alpha_n)$.

Conversely, given any family $(I_r \mid 1 \leqslant r \leqslant n)$ of proper ideals in A satisfying $I_1 \cap \dots \cap I_n = \{0\}$, there is an extension $A(u_1 \dots u_n)$ of A such that $I_r = \{a \in A \mid a \cdot u_r = 0\}$: let $D = A(x_1 \dots x_n)$ be the free product of A and a finite BA with atoms $x_1, \dots, x_n$. Let

$$K = \{i_1 \cdot x_1 + \dots + i_n \cdot x_n \mid i_1 \in I_1, \dots, i_n \in I_n\}.$$

K is an ideal of D; the canonical epimorphism π from D onto $B = D/K$ is one- one on A, and for $a \in A$, $\pi(a) \cdot u_r = 0$ iff $a \in I_r$ where $u_r = \pi(x_r)$. Now identify A with the subalgebra $\pi(A)$ of B.

For the rest of this section we think, as in section 1, of B as being the set of global sections of a sheaf $\mathcal{S} = (S, \pi, X, \mu)$ of Boolean algebras over a