# 1. The sheaf representation of Boolean algebra extensions

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## 1. The sheaf representation of Boolean algebra extensions

Let  $\mathscr{L}$  be any language for first-order predicate logic. Suppose X is a non-empty set and for every  $p \in X$  we have an  $\mathscr{L}$ -structure  $\mathscr{B}_p = (B_p, ...)$ ; put  $S = \bigcup_{p \in X} B_p$ . Suppose  $\varphi(x_1 ... x_n)$  is an  $\mathscr{L}$ -formula,  $u \subseteq X$  and  $f_1, ..., f_n : u \to S$  are such that  $f_i(p) \in B_p$  for  $1 \le i \le n$  and  $p \in u$ . Then let

$$\| \varphi [f_1 ... f_n] \| = \{ p \in u \mid \mathcal{B}_p \mid = \varphi [f_1 (p) ... f_n (p)] \}.$$

We may think of  $\| \varphi [f_1 ... f_n] \| \subseteq X$  as being a (Boolean) truth value of  $\varphi [f_1 ... f_n]$  in the power set of X.

A sheaf of  $\mathcal{L}$ -structures is a sequence

$$\mathcal{S} = (S, \pi, X, \mu)$$

such that a) S and X are topological spaces and  $\pi: S \to X$  is a continuous open local homeomorphism from S onto X, b)  $\mu$  is a function assigning to each  $p \in X$  an  $\mathcal{L}$ -structure  $\mathcal{B}_p = (B_p, ...)$  such that the  $B_p$  are pairwise disjoint,  $S = \bigcup_{p \in X} B_p$  and  $\pi(s) = p$  iff  $s \in B_p$ , c) for every open subset u of X and continuous  $f_1, ..., f_n: u \to S$  satisfying  $f_i(p) \in B_p$  for  $p \in u$  and every atomic  $\mathcal{L}$ -formula  $\varphi(x_1 ... x_n)$ ,  $\|\varphi[f_1 ... f_n]\|$  is an open subset of u.

The  $\mathcal{L}$ -structure  $\mathcal{B}_p$  is called the stalk of  $\mathcal{L}$  over p. — Let, if  $\mathcal{L}$  is a sheaf of  $\mathcal{L}$ -structures,  $\Gamma(\mathcal{L})$  be the set of all continuous functions  $f: X \to S$  satisfying  $f(p) \in \mathcal{B}_p$  for  $p \in X$  (the set of "global sections" of  $\mathcal{L}$ ). So  $\Gamma(\mathcal{L})$  is, if non-empty, (the underlying set of) a substructure of the product structure  $\prod_{p \in X} \mathcal{B}_p$ , hence an  $\mathcal{L}$ -structure.

For the rest of the paper, let  $\mathcal{L} = \{+, \cdot, -, 0, 1, U\}$  where U is a unary predicate. We indicate how, for a given BA extension (B, A), B may be represented by  $\Gamma$  ( $\mathcal{S}$ ) where  $\mathcal{S}$  is a sheaf of  $\mathcal{L}$ -structures over a Boolean space. We omit most of the proofs which are easy and entirely analoguous to well-known representation theorems for lattices over Boolean spaces. Let X be the Stone space of A, i.e. the set of all ultrafilters of A with the usual topology. For  $p \in X$ , let be the filter of B generated by p. Let  $\pi_p: B \to B/ = B_p$  be the canonical epimorphism. So  $B_p$  is a BA with at least two elements. For  $p, q \in X$  and  $p \neq q$ ,  $B_p$  and  $B_q$  are disjoint. Let  $S = \bigcup_{p \in X} B_p$  and  $\pi: S \to X$  be defined as stated in b) above. Let, for  $p \in X$ ,  $\mu(p)$  be the  $\mathcal{L}$ -structure  $(B_p, ..., \{0, 1\})$ . For  $u \subseteq X$  open and  $b \in B$ , let  $M_{ub} = \{\pi_p(b) \mid p \in u\}$ . The set of these  $M_{ub}$  constitutes a base

for a topology of S, and this makes  $\mathscr{S} = (S, \pi, X, \mu)$  a sheaf of  $\mathscr{L}$ -structures. Furthermore, for  $b \in B$ ,  $\sigma_b : X \to S$  defined by  $\sigma_b(p) = \pi_p(b)$  is a global section of  $\mathscr{S}$  and

$$\begin{array}{c}
\sigma: B \to \Gamma\left(\mathcal{S}\right) \\
b \mapsto \sigma_{b}
\end{array}$$

is an isomorphism from B onto  $\Gamma$  ( $\mathcal{S}$ ). We shall now identify B with  $\Gamma$  ( $\mathcal{S}$ ); so every  $b \in B$  is a function from X to S. This identifies A with those  $b \in B$  such that for every  $p \in X$  b (p) = 0 or b (p) = 1, i.e. with those  $b \in B$  satisfying ||U(b)|| = X. Let C be the BA of clopen subsets of X and e (c) the characteristic function of c for  $c \in C$ . Thus e is an isomorphism from C onto  $A \subseteq B$ .

In the rest of this section, we show that the property of being a Hausdorff sheaf for  $\mathcal S$  is equivalent to a property of the extension (B,A) which reflects, in a way which is first-order expressible in  $\mathcal S$ , completeness of the embedding of A into B. Recall that, for a sheaf  $\mathcal S$  over a Boolean space X, S is a  $T_2$ -space iff, for any  $f,g\in\Gamma(\mathcal S)$ ,  $\|f=g\|$  is a clopen subset of X;  $\mathcal S$  is then said to be a Hausdorff sheaf. Call A relatively complete in B if, for every  $b\in B$ , there is a largest element  $a\in A$  such that  $a\leqslant b$ , equivalently: for  $b\in B$ , there is a largest  $a\in A$  such that  $a\cdot b=0$  or: for  $b\in B$ , there is a smallest  $a\in A$  such that  $b\leqslant a$ .

1.1. Proposition.  $\mathcal{G}$  is a Hausdorff sheaf iff A is relatively complete in B.

*Proof.* Suppose  $\mathcal{S}$  is Hausdorff and  $b \in B$ . Let  $d \in B$  such that d(p) = 0 for every  $p \in X$ , let  $c = \|b = d\|$  and a = e(c). Then a is the largest element of A satisfying  $a \cdot b = 0$ .

Conversely, let A be relatively complete in B and suppose  $f, g \in B$ . Let a be the largest element of A such that  $a \leqslant f \cdot g + -f \cdot -g$ . Let  $c \in C$  such that a = e(c). Then ||f = g|| = c is a clopen subset of X.

1.2. Remark. Let A be relatively complete in B. Then the inclusion map from A to B is a complete homomorphism.

*Proof.* Suppose M is a subset of A having a supremum a in A. We show that a is the supremum of M in B. Clearly, a is an upper bound for M in B. Suppose that b is another upper bound for M in B. Let  $\alpha \in A$  be the largest element of A such that  $\alpha \leqslant b$ . For every  $m \in M \subseteq A$ , we have  $m \leqslant b$ , hence  $m \leqslant \alpha$  and  $a \leqslant \alpha \leqslant b$ .

The following facts are trivial:

- 1.3. Remark. a) Let A and the inclusion map from A to B be complete. Then A is relatively complete in B.
- b) Suppose A is relatively complete in B and B is complete. Then A is complete.

### 2. Relative automorphisms of finite extensions

We first give an internal description of a finite extension (B, A) where  $B = A(u_1 \dots u_n)$  and  $n \in \omega$ . We shall always assume that  $u_1, \dots, u_n$  are the atoms of the subalgebra of B generated by  $u_1, \dots, u_n$ ; i.e. that they are non-zero, pairwise disjoint and  $u_1 + \dots + u_n = 1$ . Let  $I_r = \{a \in A \mid a \cdot u_r = 0\}$  for  $1 \le r \le n$ . Clearly, each  $I_r$  is a proper ideal of A and  $I_1 \cap \dots \cap I_n = \{0\}$ . The family  $(I_r \mid 1 \le r \le n)$  completely characterizes the extension (B, A):

2.1. Remark. Suppose  $C = A(v_1 \dots v_n)$  is a finite extension of A where  $v_1, \dots, v_n$  are pairwise disjoint and  $1 = v_1 + \dots + v_n$ . Let  $B = A(u_1 \dots u_n)$  be as above. There is an isomorphism g from B onto C satisfying g(a) = a for  $a \in A$  and  $g(u_r) = v_r$  iff, for each r,  $\{a \in A \mid a \cdot v_r = 0\} = I_r$ .

Proof. By Theorem 12.4 in [7].

2.2. Remark. A is relatively complete in  $B = A(u_1 \dots u_n)$  iff, for each r,  $I_r$  is a principal ideal.

*Proof.* The only—if part follows by the definition of relative completeness. Now suppose  $\alpha_r \in A$  generates  $I_r$ ; let  $b \in B$  and  $I = \{a \in A \mid a \cdot b = 0\}$ . There are  $a_1, ..., a_n \in A$  such that  $b = a_1 \cdot u_1 + ... + a_n \cdot u_n$ . It follows that I is the principal ideal generated by  $\alpha = (-a_1 + \alpha_1) \cdot ... \cdot (-a_n + \alpha_n)$ .

Conversely, given any family  $(I_r \mid 1 \le r \le n)$  of proper ideals in A satisfying  $I_1 \cap ... \cap I_n = \{0\}$ , there is an extension  $A(u_1 ... u_n)$  of A such that  $I_r = \{a \in A \mid a \cdot u_r = 0\}$ : let  $D = A(x_1 ... x_n)$  be the free product of A and a finite BA with atoms  $x_1, ..., x_n$ . Let

$$K = \{i_1 \cdot x_1 + ... + i_n \cdot x_n \mid i_1 \in I_1, ..., i_n \in I_n\}.$$

K is an ideal of D; the canonical epimorphism  $\pi$  from D onto B = D/K is one- one on A, and for  $a \in A$ ,  $\pi(a) \cdot u_r = 0$  iff  $a \in I_r$  where  $u_r = \pi(x_r)$ . Now identify A with the subalgebra  $\pi(A)$  of B.

For the rest of this section we think, as in section 1, of B as being the set of global sections of a sheaf  $\mathcal{S} = (S, \pi, X, \mu)$  of Boolean algebras over a