3. Properties of the third product

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **31 (1985)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **26.05.2024**

$(\alpha\mathbf{A})_p \in \operatorname{End} \wedge^p V$ be the p^{th} component of $\alpha\mathbf{A} \in \amalg_p \operatorname{End} \wedge^p V$. Then $(\alpha\mathbf{A})_n \in \operatorname{End} \wedge^n V$ is scalar multiplication by a unique element of R.

2.2 *Definition:* If V is a traceable module of rank $n > 0$ over a commutative ground ring R with unit, the *trace* of any $\mathbf{A} \in \amalg_p \operatorname{End} \wedge^p V$ is the unique element $\operatorname{tr} \mathbf{A} \in R$ such that $(\alpha\mathbf{A})_n = (\operatorname{tr} \mathbf{A})I_n \in \operatorname{End} \wedge^n V$, for the identity endomorphism $I_n \in \operatorname{End} \wedge^n V$.

For example, if $A \in \operatorname{End} V$ then $(\alpha A)_n = A \cdot I_{n-1}$ for the identity endomorphism $I_{n-1} \in \operatorname{End} \wedge^{n-1} V$. One easily verifies that if V is a free R-module of rank n then the classical trace of A is precisely that element $\operatorname{tr} A \in R$ such that $A \cdot I_{n-1} = (\operatorname{tr} A)I_n \in \operatorname{End} \wedge^n V$.

2.3 THEOREM. *Let* $\amalg_p \operatorname{End} \wedge^p V$ *be the endomorphism algebra generated by the endomorphisms of a traceable module* V, *multiplication being the third product; then the trace is an algebra homomorphism* $\amalg_p \operatorname{End} \wedge^p V \xrightarrow{\operatorname{tr}} R$ *over the ground ring* R. *Specifically, both* $\operatorname{tr}(\mathbf{A}+\mathbf{B}) = \operatorname{tr} \mathbf{A} + \operatorname{tr} \mathbf{B}$ *and* $\operatorname{tr}(\mathbf{A} \times \mathbf{B}) = (\operatorname{tr} \mathbf{A})(\operatorname{tr} \mathbf{B})$ *for any elements* $\mathbf{A}$ *and* $\mathbf{B}$ *of* $\amalg_p \operatorname{End} \wedge^p V$.

Proof. Additivity of the trace is trivial. To show that the trace also respects the third product suppose that V is traceable of rank n, and let $(\alpha\mathbf{A})_p, (\alpha\mathbf{B})_p$ and $\alpha(\mathbf{A} \times \mathbf{B})_p$ denote the components of $\alpha\mathbf{A}$, $\alpha\mathbf{B}$ and $\alpha(\mathbf{A} \times \mathbf{B})$ in $\operatorname{End} \wedge^p V$ for each $p = 0, \ldots, n$. By the definition $\mathbf{A} \times \mathbf{B} = \alpha^{-1}((\alpha\mathbf{A})(\alpha\mathbf{B}))$ of the third product one has $\alpha(\mathbf{A} \times \mathbf{B}) = (\alpha\mathbf{A})(\alpha\mathbf{B})$ for the composition product $(\alpha\mathbf{A})(\alpha\mathbf{B})$, that is, $\amalg_p \alpha(\mathbf{A} \times \mathbf{B})_p = \amalg_p(\alpha\mathbf{A})_p(\alpha\mathbf{B})_p$. In particular $\alpha(\mathbf{A} \times \mathbf{B})_n = (\alpha\mathbf{A})_n(\alpha\mathbf{B})_n$ in the n^{th} component $\operatorname{End} \wedge^n V$, so that

$$\operatorname{tr}(\mathbf{A} \times \mathbf{B})I_n = ((\operatorname{tr} \mathbf{A})I_n)((\operatorname{tr} \mathbf{B})I_n) = (\operatorname{tr} \mathbf{A})(\operatorname{tr} \mathbf{B})I_n$$

by definition of the trace; since $\operatorname{End} \wedge^n V$ is free on the single generator I_n this implies $\operatorname{tr}(\mathbf{A} \times \mathbf{B}) = (\operatorname{tr} \mathbf{A})(\operatorname{tr} \mathbf{B})$ as claimed.

3. PROPERTIES OF THE THIRD PRODUCT

We now establish several properties of the third product. Although these properties do not require the R-module V to be traceable, we shall later impose a condition on elements of the R-module $\Pi_r \operatorname{End} \wedge^r V$ itself; the condition will automatically be satisfied in the applications.

Let V be any module over a commutative ring R with unit, and let $\mathbf{A}$ and $\mathbf{B}$ be elements of the direct product $\Pi_r \operatorname{End} \wedge^r V$ whose only nonvanishing components occur in degrees p and q, respectively; for convenience we write A_p and B_q in place of $\mathbf{A}$ and $\mathbf{B}$. Recall that the identity endomorphism of each exterior power $\wedge^r V$ is given by $I_r = \frac{1}{r!} I^{\cdot r} \in \operatorname{End} \wedge^r V$ for the identity endomorphism $I = I_1 \in \operatorname{End} V$, and that there is a two-sided unit element $\mathbf{I} = e^{\cdot I} \in \Pi_r \operatorname{End} \wedge^r V$ with respect to composition products.

3.1 LEMMA. *For each* $r \geqslant 0$ *let* $(A_p \times B_q)_r \in \operatorname{End} \wedge^r V$ *be the* r^{th} *component of* $A_p \times B_q$; *then*

$$(A_p \times B_q)_r = \sum_s (-1)^s (A_p \cdot I_{r-s-p})(B_q \cdot I_{r-s-q}) \cdot I_s ,$$

where $I_t = 0$ *for* $t < 0$.

Proof. This is an immediate consequence of the definition

$$\mathbf{A} \times \mathbf{B} = \alpha^{-1}((\alpha\mathbf{A})(\alpha\mathbf{B})), \qquad \text{where} \qquad \alpha\mathbf{A} = e^{\cdot I} \cdot \mathbf{A}, \alpha\mathbf{B} = e^{\cdot I} \cdot \mathbf{B},$$

and $\alpha^{-1}\mathbf{C} = e^{\cdot(-I)} \cdot \mathbf{C}$.

3.2 LEMMA. $(A_p \times B_q)_r = 0$ *for* $r < \max(p, q)$, *and if* $p = q = r$ *then* $(A_r \times B_r)_r$ *is the composition* $A_r B_r \in \operatorname{End} \wedge^r V$.

Proof. Immediate consequence of Lemma 3.1.

One can probably also use Lemma 3.1 directly to obtain the following more interesting properties of the third product: $(A_p \times B_q)_r = 0$ for $r > p + q$, and $(A_p \times B_q)_{p+q}$ is the shuffle product $A_p \cdot B_q \in \operatorname{End} \wedge^{p+q} V$. However, in order to avoid cumbersome computations we prove these results only for somewhat restricted endomorphisms $A_p \in \operatorname{End} \wedge^p V$ and $B_q \in \operatorname{End} \wedge^q V$.

3.3 *Definition:* An element $\mathbf{A} \in \Pi_r \operatorname{End} \wedge^r V$ is *one-generated* whenever each component is an R-linear combination of shuffle products of endomorphisms of V itself.

Clearly sums and all products of one-generated elements are one-generated; thus the one-generated elements form a subalgebra of $\Pi_r \operatorname{End} \wedge^r V$, with respect to any of the three products.

3.4 LEMMA. *For any one-generated elements* $\mathbf{B}$ *and* $\mathbf{C}$ *of* $\Pi_r \operatorname{End} \wedge^r V$ *and any* $A \in \operatorname{End} V$ *one has* $(\alpha A)(\mathbf{B} \cdot \mathbf{C}) = (\alpha A)\mathbf{B} \cdot \mathbf{C} + \mathbf{B} \cdot (\alpha A)\mathbf{C}$.

Proof. One may as well choose $\mathbf{B}$ and $\mathbf{C}$ to be shuffle products

$$B_1 \cdot ... \cdot B_p \in \operatorname{End} \wedge^p V \quad \text{and} \quad C_1 \cdot ... \cdot C_q \in \operatorname{End} \wedge^q V$$

of endomorphisms $B_1, ..., B_p, C_1, ..., C_q$ of V itself. Then

$$(\alpha A)\mathbf{B} = (A \cdot I_{p-1})(B_1 \cdot ... \cdot B_p) = \sum_{s=1}^{p} B_1 \cdot ... \cdot AB_s \cdot ... \cdot B_p,$$

and the result follows from the observation that $(\alpha A)\mathbf{C}$ and $(\alpha A)(\mathbf{B} \cdot \mathbf{C})$ are similar sums of shuffle products.

3.5 Lemma. *For any* $A \in \operatorname{End} V$ *and any one-generated* $\mathbf{C} \in \Pi_r \operatorname{End} \wedge^r V$ *one has* $A \times \mathbf{C} = A \cdot \mathbf{C} + (\alpha A)\mathbf{C}$.

Proof. Suppose that $\mathbf{C} \in \operatorname{End} \wedge^q V$, and use subscripts r to identify components of $\operatorname{End} \wedge^r V$. Then Lemma 3.4 yields

$$\begin{aligned}(\alpha A)(\alpha \mathbf{C})_r &= (\alpha A)(I_{r-q} \cdot \mathbf{C}) = (\alpha A)I_{r-q} \cdot \mathbf{C} + I_{r-q} \cdot (\alpha A)\mathbf{C} \\ &= (A \cdot I_{r-q-1}) \cdot \mathbf{C} + I_{r-q} \cdot (\alpha A)\mathbf{C} \\ &= I_{r-q-1} \cdot (A \cdot \mathbf{C}) + I_{r-q} \cdot (\alpha A)\mathbf{C} = \alpha(A \cdot \mathbf{C})_r + \alpha((\alpha A)\mathbf{C})_r,\end{aligned}$$

hence

$$(\alpha A)(\alpha \mathbf{C}) = \alpha(A \cdot \mathbf{C} + (\alpha A)\mathbf{C}) \in \Pi_r \operatorname{End} \wedge^r V,$$

hence

$$A \times \mathbf{C} = \alpha^{-1}((\alpha A)(\alpha \mathbf{C})) = A \cdot \mathbf{C} + (\alpha A)\mathbf{C}$$

as claimed.

3.6 Lemma. *For any one-generated elements* $\mathbf{A} = A_p \in \operatorname{End} \wedge^p V$ *and* $\mathbf{B} = B_q \in \operatorname{End} \wedge^q V$ *one has* $(A_p \times B_q)_r = 0 \in \operatorname{End} \wedge^r V$ *for* $r > p + q$ *and* $(A_p \times B_q)_{p+q} = A_p \cdot B_q \in \operatorname{End} \wedge^{p+q} V$.

Proof. Let $J_{p+q-1} \subset \Pi_r \operatorname{End} \wedge^r V$ be the R-submodule consisting of the summands $\operatorname{End} \wedge^r V$ for $r < p + q$. It suffices to show by induction on p that $A_p \times B_q - A_p \cdot B_q \in J_{p+q-1}$, the case $p = 0$ being trivial. One may as well assume that $A_p = A_1 \cdot A_{p-1}$ for $A_1 \in \operatorname{End} V$ and a one-generated element $A_{p-1} \in \operatorname{End} \wedge^{p-1} V$. Then

$$A_p \times B_q = (A_1 \cdot A_{p-1}) \times B_q = (A_1 \times A_{p-1}) \times B_q - (\alpha A_1)A_{p-1} \times B_q$$

by Lemma 3.5, where $(\alpha A_1)A_{p-1} \times B_q \in J_{p+q-1}$ by a weak form of the inductive hypothesis. One also has $A_{p-1} \times B_q \in J_{p+q-1}$ by the same weak form of the inductive hypothesis, so that $(\alpha A_1)(A_{p-1} \times B_q) \in J_{p+q-1}$; hence a second application of Lemma 3.5 gives

$$A_p \times B_q = (A_1 \times A_{p-1}) \times B_q \bmod J_{p+q-1} = A_1 \times (A_{p-1} \times B_q) \bmod J_{p+q-1}$$
$$= A_1 \cdot (A_{p-1} \times B_q) \bmod J_{p+q-1} .$$

Finally, the specific form $A_{p-1} \times B_q - A_{p-1} \cdot B_q \in J_{p+q-2} (\subset J_{p+q-1})$ of the inductive hypothesis permits one to conclude that

$$A_p \times B_q = A_1 \cdot (A_{p-1} \cdot B_q) \bmod J_{p+q-1} = (A_1 \cdot A_{p-1}) \cdot B_q \bmod J_{p+q-1}$$
$$= A_p \cdot B_q \bmod J_{p+q-1} ,$$

which completes the inductive step.

3.7 PROPOSITION. *For any module* V *over a commutative ring* R *with unit, the third product in* $\Pi_r \operatorname{End} \wedge^r V$ *restricts to a product in the submodule of one-generated elements of the direct sum* $\amalg_r \operatorname{End} \wedge^r V$.

Proof. Immediate consequence of Lemma 3.6.

Lemma 3.6 and Proposition 3.7 are certainly valid under considerably weaker hypotheses; for example, one can easily combine the present versions with localization techniques to obtain greater generality. One can possibly establish entirely unrestricted versions of Lemma 3.6 and Proposition 3.7 by applying the identity of Lemma 3.1 directly to elements

$$x_1 \wedge \dots \wedge x_r \in \wedge^r V \quad \text{for} \quad r \geqslant p + q ;$$

however, such a computation would probably be very complicated.

Here is a simple application of the results of this section. Any endomorphisms $A \in \operatorname{End} V$ and $B \in \operatorname{End} V$ are trivially one-generated, so that Lemmas 3.2 and 3.6 imply $A \times B = AB + A \cdot B$ and $B \times A = BA + B \cdot A$; since $A \cdot B = B \cdot A$ it follows that $AB - BA = A \times B - B \times A$. If V is traceable Theorem 2.3 then implies the best-known elementary property of the trace:

$$\operatorname{tr} AB - \operatorname{tr} BA = (\operatorname{tr} A)(\operatorname{tr} B) - (\operatorname{tr} B)(\operatorname{tr} A) = 0 .$$

4. NEWTON IDENTITIES

Let A be any endomorphism of a module V over a commutative ring R with unit. The shuffle products of the compositions $I, A, A^2, \dots$ of A