NOTE ON LEVI'S PROBLEM WITH DISCONTINUOUS FUNCTIONS

L'Enseignement Mathématique, t. 31 (1985), p. 299-304

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§ 1. Introduction

In [3] Fornaess and Narasimhan proved that a complex space X which carries a strongly plurisubharmonic exhaustion function $\varphi : X \to \mathbf{R}$ is a Stein space. It is a remarkable fact that φ is supposed only upper semicontinuous.

A natural question which arises when we consider the Levi problem with upper semicontinuous functions is the following: what would happen if we allowed φ to take on the value $-\infty$. Simple examples (compact complex spaces, the blowing up of $\mathbf{C}^n$ at the origine...) show us that X is not necessarily Stein. The best result one might hope to obtain is X being 1-convex.

The aim of this short note is to give an affirmative answer to this question, hence to prove the following theorem conjectured by Fornaess and Narasimhan:

Theorem 1. *Let X be a complex space which admits a strongly plurisubharmonic exhaustion function $\varphi : X \to [-\infty, \infty)$. Then X is 1-convex.*

If φ is supposed real-valued it follows easily, from the maximum principle, that the exceptional set of X is empty, hence X is Stein. This is exactly Fornaess-Narasimhan's theorem.

§ 2. Preliminaries

All complex spaces are assumed to be reduced and countable at infinity.

An upper semicontinuous function $\varphi : X \to [-\infty, \infty)$ is called plurisubharmonic if for every holomorphic map $\tau : W \to X$ (W = the unit disc in $\mathbf{C}$) it follows that $\varphi \circ \tau$ is subharmonic on W (possibly $\equiv -\infty$). φ is said to be strongly plurisubharmonic if for every C^∞ real-valued function θ with compact support there exists an $\varepsilon_0 > 0$ such that $\varphi + \varepsilon\theta$ is plurisubharmonic for $|\varepsilon| \leqslant \varepsilon_0$.

A main result in [3] tells us that the above definition agrees with the usual one as given in [6].

Let us also recall that a complex space X is said to be 1-convex if there exist:

i) a compact analytic set $S \subset X$ with $\dim_x S > 0$ for any $x \in S$,

ii) a Stein space Y, a finite set $A \subset Y$ and a proper holomorphic map $p: X \to Y$ inducing a bilholomorphism $X \backslash S \cong Y \backslash A$ and which satisfies $p_* \mathcal{O}_X \cong \mathcal{O}_Y$.

S is called the exceptional set of X and Y the Remmert reduction of X.

Remark. Using the analytic version of Chow's lemma (Hironaka [5]) it was proved in [2] that any 1-convex space X carries a strongly plurisubharmonic exhaustion function $\varphi: X \to [-\infty, \infty)$, i.e. the converse of Theorem 1 holds too.

§ 3. The proof of Theorem

We shall apply Andreotti-Grauert's technique [1] with suitable modifications required by the upper semicontinuity. Throughout this section $\mathcal{F}$ will denote a coherent sheaf on X and $X_c = \{x \in X \mid \varphi(x) < c\}$.

To prove Theorem 1 we need some lemmas.

Lemma 1. *For any $c \in \mathbf{R}$ there exists $\varepsilon > 0$ such that is restriction map $H^1(X_{c+\varepsilon}, \mathcal{F}) \to H^1(X_{c+\varepsilon'}, \mathcal{F})$ is surjective for any $0 \leqslant \varepsilon' \leqslant \varepsilon$.*

Proof. We may assume $c = 0$. Set $K = \overline{\{\varphi < 1\}}$ and let $\{U_1, \ldots, U_m\}$ be a covering of K with Stein open sets, $U_i \subset\subset X$ and $h_i \in C_0^\infty(U_i)$, $h_i \geqslant 0$ such that $\varphi - \sum_{i=1}^{r} h_i$ is strongly plurisubharmonic for $r = 1, \ldots, m$ and $\sum_{i=1}^{m} h_i > 0$ on K. Choose $\alpha > 0$ such that $\sum_{i=1}^{m} h_i(x) \geqslant \alpha$ for any $x \in K$ and take $0 < \varepsilon < \min(\alpha, 1)$. We shall prove that this ε satisfies the conditions required in Lemma 1.

For any $0 \leqslant \varepsilon' \leqslant \varepsilon$ we set $X_{\varepsilon'}^r = \{x \in X \mid \varphi(x) < \varepsilon' + h_1(x) + \ldots + h_r(x)\}$ for $r = 0, \ldots, m$ (by definition $X_{\varepsilon'}^0 = X_{\varepsilon'}$).

We make the following remark: for any $0 \leqslant \varepsilon' \leqslant \varepsilon$ we have $X_\varepsilon \subset X_{\varepsilon'}^m$. Indeed, let $x \in X$ such that $\varphi(x) < \varepsilon$. In particular $\varphi(x) < 1$, hence $x \in K$. From the definition of α it follows that $\sum_{i=1}^m h_i(x) \geqslant \alpha$ and from the inequalities $\varphi(x) < \varepsilon < \alpha \leqslant \sum_{i=1}^m h_i(x) \leqslant \varepsilon' + \sum_{i=1}^m h_i(x)$ we get $x \in X_{\varepsilon'}^m$.

Due to this remark Lemma 1 will be proved if we prove that the restriction map $H^1(X_{\varepsilon'}^m, \mathscr{F}) \to H^1(X_{\varepsilon'}, \mathscr{F})$ is surjective for any $0 \leqslant \varepsilon' \leqslant \varepsilon$. The inclusions $X_{\varepsilon'} = X_{\varepsilon'}^0 \subset X_{\varepsilon'}^1 \subset \dots \subset X_{\varepsilon'}^m$ show that it suffices to prove that the restrictions $H^1(X_{\varepsilon'}^{r+1}, \mathscr{F}) \to H^1(X_{\varepsilon'}^r, \mathscr{F})$ are surjective for $r = 0, \dots, m-1$. If we set

$$V_{\varepsilon'}^{r+1} = \{x \in U_{r+1} \mid \varphi(x) < \varepsilon' + h_1(x) + \dots + h_{r+1}(x)\}$$

then $V_{\varepsilon'}^{r+1}$ and $X_{\varepsilon'}^r \cap V_{\varepsilon'}^{r+1}$ are Stein open sets. On the other hand $X_{\varepsilon'}^{r+1} \backslash X_{\varepsilon'}^r \subset \operatorname{supp}(h_{r+1}) \subset U_{r+1}$ and so $X_{\varepsilon'}^{r+1} = X_{\varepsilon'}^r \cup V_{\varepsilon'}^{r+1}$. From the Mayer-Vietoris exact sequence:

$$H^1(X_{\varepsilon'}^{r+1}, \mathscr{F}) \to H^1(X_{\varepsilon'}^r, \mathscr{F}) \oplus H^1(V_{\varepsilon'}^{r+1}, \mathscr{F}) \to H^1(X_{\varepsilon'}^r \cap V_{\varepsilon'}^{r+1}, \mathscr{F})$$

it follows that the restriction map $H^1(X_{\varepsilon'}^{r+1}, \mathscr{F}) \to H^1(X_{\varepsilon'}^r, \mathscr{F})$ is surjective and so Lemma 1 is proved.

LEMMA 2. *For any* $\alpha \leqslant \beta$ *the restriction map* $H^1(X_\beta, \mathscr{F}) \to H^1(X_\alpha, \mathscr{F})$ *is surjective.*

Proof. Set $M(\alpha) = \{\delta \geqslant \alpha \mid$ for any $\alpha \leqslant \gamma \leqslant \delta$ the restriction map

$$H^1(X_\delta, \mathscr{F}) \to H^1(X_\gamma, \mathscr{F}) \text{ is surjective}\}.$$

From Lemma 1 and Lemma [1, p. 241] we deduce that $M(\alpha) = [\alpha, \infty)$ which proves Lemma 2.

LEMMA 3. *For any* $\alpha \in \mathbf{R}$ $H^1(X_\alpha, \mathscr{F})$ *has finite dimension.*

Proof. Choose $\beta > \alpha$ such that $\bar{X}_\alpha \subset X_\beta$. From Lemma 2 the restriction map $H^1(X_\beta, \mathscr{F}) \to H^1(X_\alpha, \mathscr{F})$ is surjective and from [1, p. 240]

$$\dim_{\mathbf{C}} H^1(X_\alpha, \mathscr{F}) < \infty .$$

LEMMA 4. *For any* $c \in \mathbf{R}$ *there exists* $\varepsilon > 0$ *such that the restriction map* $\Gamma(X_{c+\varepsilon}, \mathscr{F}) \to \Gamma(X_{c+\varepsilon'}, \mathscr{F})$ *has dense image for any* $0 \leqslant \varepsilon' \leqslant \varepsilon$.

Proof. We may assume $c = 0$ and choose $\varepsilon > 0$ as in Lemma 1. Exactly as in the proof of Lemma 1 it suffices to prove that the restriction map $\Gamma(X_{\varepsilon'}^{r+1}, \mathscr{F}) \to \Gamma(X_{\varepsilon'}^{r}, \mathscr{F})$ has dense image for $r = 0, \dots, m-1$.

Consider the Mayer-Vietoris exact sequence:

$$\Gamma(X_{\varepsilon'}^{r+1}, \mathscr{F}) \to \Gamma(X_{\varepsilon'}^{r}, \mathscr{F}) \oplus \Gamma(V_{\varepsilon'}^{r+1}, \mathscr{F}) \xrightarrow{\alpha} \Gamma(X_{\varepsilon'}^{r} \cap V_{\varepsilon'}^{r+1}, \mathscr{F}) \to H^1(X_{\varepsilon'}^{r+1}, \mathscr{F})$$

Since $(X_{\varepsilon'}^{r} \cap V_{\varepsilon'}^{r+1}, V_{\varepsilon'}^{r+1})$ is a Runge pair it follows that α has dense image. On the other hand, applying Lemma 3 to the function

$$\varphi - \varepsilon' - h_1 - \dots - h_{r+1}$$

we deduce that $H^1(X_{\varepsilon'}^{r+1}, \mathscr{F})$ has finite dimension, in particular it is separated, hence α has closed image. Consequently α is surjective. From the open mapping theorem it follows easily that the restriction map

$$\Gamma(X_{\varepsilon'}^{r+1}, \mathscr{F}) \to \Gamma(X_{\varepsilon'}^{r}, \mathscr{F})$$

has dense image and so Lemma 4 is proved.

LEMMA 5. *For any $\alpha \leqslant \beta$ the restriction map $\Gamma(X_\beta, \mathscr{F}) \to \Gamma(X_\alpha, \mathscr{F})$ has dense image.*

Proof. Lemma 5 is an immediate consequence of Lemma 4 and of Lemma [1, p. 246].

LEMMA 6. *For any $c \in \mathbf{R}$ there exists $\varepsilon > 0$ such that the restriction map $H^1(X_{c+\varepsilon}, \mathscr{F}) \to H^1(X_{c+\varepsilon'}, \mathscr{F})$ is bijective for any $0 \leqslant \varepsilon' \leqslant \varepsilon$.*

Proof. We may assume $c = 0$ and choose $\varepsilon > 0$ as in Lemma 1. Due to the inclusions $X_{\varepsilon'} \subset X_\varepsilon \subset X_{\varepsilon'}^m$ and using Lemma 2 it follows that it suffices to show that the restriction map $H^1(X_{\varepsilon'}^m, \mathscr{F}) \to H^1(X_{\varepsilon'}, \mathscr{F})$ is bijective. The inclusions $X_{\varepsilon'} = X_{\varepsilon'}^0 \subset X_{\varepsilon'}^1 \subset \dots \subset X_{\varepsilon'}^m$ show that it is enough to prove that the restrictions $H^1(X_{\varepsilon'}^{r+1}, \mathscr{F}) \to H^1(X_{\varepsilon'}^{r}, \mathscr{F})$ are bijective for $r = 0, \dots, m-1$.

Consider the Mayer-Vietoris exact sequence:

$$\Gamma(X_{\varepsilon'}^{r}, \mathscr{F}) \oplus \Gamma(V_{\varepsilon'}^{r+1}, \mathscr{F}) \to \Gamma(X_{\varepsilon'}^{r} \cap V_{\varepsilon'}^{r+1}, \mathscr{F}) \to H^1(X_{\varepsilon'}^{r+1}, \mathscr{F}) \to H^1(X_{\varepsilon'}^{r}, \mathscr{F}) \oplus H^1(V_{\varepsilon'}^{r+1}, \mathscr{F}) \to H^1(X_{\varepsilon'}^{r} \cap V_{\varepsilon'}^{r+1}, \mathscr{F})$$

As remarked in the proof of Lemma 4 the map

$$\Gamma(X_{\varepsilon'}^{r}, \mathscr{F}) \oplus \Gamma(V_{\varepsilon'}^{r+1}, \mathscr{F}) \to \Gamma(X_{\varepsilon'}^{r} \cap V_{\varepsilon'}^{r+1}, \mathscr{F})$$

is surjective. Since

$$H^1(V_{\varepsilon'}^{r+1}, \mathscr{F}) = H^1(X_{\varepsilon'}^{r} \cap V_{\varepsilon'}^{r+1}, \mathscr{F}) = 0$$

it follows that the restriction map

$$H^1(X_{\varepsilon'}^{r+1}, \mathscr{F}) \to H^1(X_{\varepsilon'}^{r}, \mathscr{F})$$

is bijective and so Lemma 6 is proved.

LEMMA 7. *For any* $\alpha \leqslant \beta$ *the restriction map* $H^1(X_\beta, \mathscr{F}) \to H^1(X_\alpha, \mathscr{F})$ *is bijective.*

Proof. Set $M(\alpha) = \{\delta \geqslant \alpha \mid$ for any $\alpha \leqslant \gamma \leqslant \delta$ the restriction map

$$H^1(X_\delta, \mathscr{F}) \to H^1(X_\gamma, \mathscr{F}) \text{ is bijective}\}$$

and let $\alpha_0 = \sup M(\alpha)$.

From Lemma 2 it follows that if $\delta \in M(\alpha)$ then $[\alpha, \delta] \subset M(\alpha)$, consequently $[\alpha, \alpha_0) \subset M(\alpha)$. To prove Lemma 7 we have to show that $\alpha_0 = \infty$. Suppose that $\alpha_0 < \infty$. From Lemma 5 and Lemma [1, p. 250] we deduce that $\alpha_0 \in M(\alpha)$. From Lemma 6 there exists $\varepsilon > 0$ such that $\alpha_0 + \varepsilon \in M(\alpha)$. This contradicts the definition of α_0, and so Lemma 7 is proved.

We are now in a position to prove Theorem 1. Choose $\alpha \in \mathbf{R}$ and take $\alpha = \alpha_0 < \alpha_1 < \dots < \alpha_n < \dots$ an increasing sequence of real numbers tending to ∞. By Lemma 7 the restriction map $H^1(X_{\alpha_{n+1}}, \mathscr{F}) \to H^1(X_{\alpha_n}, \mathscr{F})$ is bijective and by Lemma 5 the restriction map $\Gamma(X_{\alpha_{n+1}}, \mathscr{F}) \to \Gamma(X_{\alpha_n}, \mathscr{F})$ has dense image. It follows then from Lemma [1, p. 250] that the restriction map $H^1(X, \mathscr{F}) \to H^1(X_\alpha, \mathscr{F})$ is also bijective and from Lemma 3 $H^1(X, \mathscr{F})$ has finite dimension. Theorem V. in [6] tells us that X is 1-convex, as required.

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(Reçu le 10 décembre 1984)

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