1. Introduction and history

HODGE DECOMPOSITION ON STRATIFIED LIE GROUPS

by John DUDDY

1. INTRODUCTION AND HISTORY

The Hodge decomposition theorem is the following:

THEOREM. *On a compact Riemannian manifold every p-form,* α, *can be written as* $\alpha = \alpha_1 + \alpha_2 + \alpha_3$ *where* $\alpha_1 = d^*\beta_1$, $\alpha_2 = d\beta_2$ *and* α_3 *is harmonic.*

This result appears in Hodge's book *The Theory and Applications of Harmonic Integrals* (1941) [12]. Since the appearance of this result generalizations of the theorem have been proven in new settings. Kodaira (1949) extended the result to certain forms on non-compact Riemannian manifolds [13] and Dolbeault (1953) derived a similar decomposition for Hermitian manifolds [5]. Atiyah and Bott (1967) defined an elliptic complex which generalized the de Rham and Dolbeault complexes [1]. In a different vein Spencer outlined a program to solve overdetermined equations (1963) [17]. The heart of his program was to obtain a Hodge decomposition paying special attention to boundary values.

Boundary value problems in complex analysis led to the $\bar{\partial}_b$ complex. It was first studied by H. Lewy (1957) [15] and generalized by Kohn and Rossi (1965) [14] and by Greenfield (1968) [10]. The complex is not elliptic but it does enjoy certain properties of elliptic complexes. For instance, its Laplacian, $\Box_b$, (with respect to a Hermitian metric) is hypoelliptic, i.e., if $\Box_b f = g$ and g is C^∞ on an open set U, then f is C^∞ in U. Folland and Stein (1973, 1974) [7, 8] wrote down an explicit fundamental solution for $\Box_b$ on the Heisenberg group. The group is not compact so Kodaira's arguments to obtain the decomposition do not apply. One of the aims of this paper is to exploit the simple homogeneity properties to obtain a fundamental solution. The technique generalizes to a class of nilpotent groups called stratified groups introduced by Folland (1975) [9]. (Also see Rothschild and Stein [16].)

The Hodge decomposition for the $\bar\partial_b$ complex on the Heisenberg group appears in [11] by Harvey and Polking and in [4]. The second reference motivates the technique used here. Harvey and Polking use complex analysis to obtain their result (solving the $\bar\partial_b$ problem first, then the $\Box_b$ problem). Using their techniques Dadok and Harvey [2] have found a fundamental solution for $\Box_b$ on the sphere in $\mathbf{C}^n$. A parametrix for $\Box_b$ on the sphere also appeared in [4] but will not be presented here, due to the more complete result of Dadok and Harvey.

Let us briefly review the Hodge decomposition. For the classical version see [3] and [12]. Let M be an n-dimensional C^∞ manifold and let E and E' be vector bundles over M whose fibers are isomorphic to $\mathbf{F}^m$ and $\mathbf{F}^{m'}$, respectively. (We let $\mathbf{F} = \mathbf{R}$ or $\mathbf{C}$.) We denote the space of smooth sections of E by $C^\infty(M, E)$ and when there is no confusion we abbreviate the notation to $C^\infty(E)$. A differential operator is a map $D\colon C^\infty(E) \to C^\infty(E')$ such that given any local trivializations of E and E' over U (where $U \subset M$ is open), D can be expressed by an $m' \times m$ matrix of differential operators defined on $\mathbf{F}$-valued functions on $\mathbf{R}^n$. See [18] for details.

Suppose we are given three vector bundles, E_1, E_2, and E_3 over M and differential operators $D_1\colon C^\infty(E_1) \to C^\infty(E_2)$ and $D_2\colon C^\infty(E_2) \to C^\infty(E_3)$. If $D_2 \circ D_1 = 0$ we say that the complex

$$C^\infty(E_1) \xrightarrow{D_1} C^\infty(E_2) \xrightarrow{D_2} C^\infty(E_3) \tag{1}$$

is a differential complex. Examples of differential complexes are the de Rham, Dolbeault, and $\bar\partial_b$ complex.

Assume there exists a measure $d\mu$ on M and a metric on the E_i which we denote by $(\cdot, \cdot)_{i,x}$ where $x \in M$. For $f, g \in C^\infty(E_i)$, one of which is compactly supported, define

$$(f, g)_i = \int_M (f(x), g(x))_{i,x} d\mu(x)\,.$$

Define the formal adjoint, D_1^*, of D_1 by the identity

$$(f, D_1 g)_2 = (D_1^* f, g)_1$$

where $f \in C^\infty(E_2)$ and $g \in C_c^\infty(E_1)$. Note that $C_c^\infty(E_i)$ is the subset of compactly supported elements of $C^\infty(E_i)$. Similarly, we define D_2^*. The Laplacian is given by

$$\Delta = D_1 D_1^* + D_2^* D_2\,.$$

Let H be the kernel of Δ in $C_c^\infty(E_2)$. A Hodge decomposition for $C_c^\infty(E_2)$ is

$$C_c^\infty(E_2) = D_1(C_c^\infty(E_1)) \oplus D_2^*(C_c^\infty(E_3)) \oplus H .$$

Hodge studied the de Rham complex on a compact Riemannian manifold. The Riemannian metric induced the metrics on the bundles $\Lambda^p T^*(M)$ as well as the volume element.

In the next section we discuss abstract CR manifolds and look at the Heisenberg group in detail. We write down the $\bar{\partial}_b$ and $\Box_b$ operators explicitly and give Folland and Stein's inverse to $\Box_b$. In section 3 we introduce the stratified Lie groups and the associated homogeneous structures. We present the continuity theorems of Folland and Rothschild and Stein for convolution operators. In section 4 we prove the decomposition theorem in the general setting of stratified groups.

These results are an extension of the author's dissertation [4]. We wish to express our deep gratitude to M. Kuranishi. We would also like to thank D. Tartakoff for his help and encouragement.

2. CR STRUCTURES AND THE HEISENBERG GROUP

Let M be a C^∞ manifold of dimension $2n + 1$. The complexified tangent bundle of M, $\mathbf{C}T(M)$, is the bundle whose fiber is $\mathbf{C} \otimes_{\mathbf{R}} T_m(M)$ where $T_m(M)$ is the tangent space at $m \in M$. When there is no confusion we will drop reference to M in the notation for $T(M)$, $\mathbf{C}T(M)$, etc. So, $T(M) = T$ and $\mathbf{C}T(M) = \mathbf{C}T$, for example.

A CR structure on M is a subbundle $T_{1,0} \subset \mathbf{C}T$ such that (i) $T_{1,0} \cap \overline{T_{1,0}} = \{0\}$, (ii) codim $(T_{1,0} \otimes \overline{T_{1,0}}) = 1$, (iii) if X and Y are smooth sections of $T_{1,0}$ then $[X, Y] = XY - YX$ is a section $T_{1,0}$. We set $T_{0,1} = \overline{T_{1,0}}$. If M has a CR structure it is called a CR manifold.

An example of a CR manifold is a real hypersurface M in a complex manifold M', $M \subset M'$. Define $T_{1,0}(M) = \mathbf{C}T(M) \cap T_{1,0}(M')$ where $T_{1,0}(M')$ is the holomorphic tangent bundle of M'.

If M is a CR manifold set $T^{1,0}$ (resp., $T^{0,1}$) to be the dual space to $T_{1,0}$ (resp., $T_{0,1}$). Let $\Lambda^{p,q}$ be the space of C^∞ sections of $\Lambda^p T^{1,0} \otimes \Lambda^q T^{0,1}$. Define the operator $\bar{\partial}_b \colon \Lambda^{p,q} \to \Lambda^{p,q+1}$ as follows: Let $\phi \in \Lambda^{p,q}$ and let $X_1, \ldots, X_p$ (resp., $Y_1, \ldots, Y_{q+1}$) be sections of $T_{1,0}$ (resp., $T_{0,1}$). Then