2. CR STRUCTURES AND THE HEISENBERG GROUP

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Let H be the kernel of Δ in $C_c^\infty(E_2)$. A Hodge decomposition for $C_c^\infty(E_2)$ is

$$C_c^\infty(E_2) = D_1(C_c^\infty(E_1)) \oplus D_2^*(C_c^\infty(E_3)) \oplus H .$$

Hodge studied the de Rham complex on a compact Riemannian manifold. The Riemannian metric induced the metrics on the bundles $\Lambda^p T^*(M)$ as well as the volume element.

In the next section we discuss abstract CR manifolds and look at the Heisenberg group in detail. We write down the $\bar{\partial}_b$ and $\Box_b$ operators explicitly and give Folland and Stein's inverse to $\Box_b$. In section 3 we introduce the stratified Lie groups and the associated homogeneous structures. We present the continuity theorems of Folland and Rothschild and Stein for convolution operators. In section 4 we prove the decomposition theorem in the general setting of stratified groups.

These results are an extension of the author's dissertation [4]. We wish to express our deep gratitude to M. Kuranishi. We would also like to thank D. Tartakoff for his help and encouragement.

2. CR STRUCTURES AND THE HEISENBERG GROUP

Let M be a C^∞ manifold of dimension $2n + 1$. The complexified tangent bundle of M, $\mathbf{C}T(M)$, is the bundle whose fiber is $\mathbf{C} \otimes_{\mathbf{R}} T_m(M)$ where $T_m(M)$ is the tangent space at $m \in M$. When there is no confusion we will drop reference to M in the notation for $T(M)$, $\mathbf{C}T(M)$, etc. So, $T(M) = T$ and $\mathbf{C}T(M) = \mathbf{C}T$, for example.

A CR structure on M is a subbundle $T_{1,0} \subset \mathbf{C}T$ such that (i) $T_{1,0} \cap \overline{T_{1,0}} = \{0\}$, (ii) codim $(T_{1,0} \otimes \overline{T_{1,0}}) = 1$, (iii) if X and Y are smooth sections of $T_{1,0}$ then $[X, Y] = XY - YX$ is a section $T_{1,0}$. We set $T_{0,1} = \overline{T_{1,0}}$. If M has a CR structure it is called a CR manifold.

An example of a CR manifold is a real hypersurface M in a complex manifold M', $M \subset M'$. Define $T_{1,0}(M) = \mathbf{C}T(M) \cap T_{1,0}(M')$ where $T_{1,0}(M')$ is the holomorphic tangent bundle of M'.

If M is a CR manifold set $T^{1,0}$ (resp., $T^{0,1}$) to be the dual space to $T_{1,0}$ (resp., $T_{0,1}$). Let $\Lambda^{p,q}$ be the space of C^∞ sections of $\Lambda^p T^{1,0} \otimes \Lambda^q T^{0,1}$. Define the operator $\bar{\partial}_b : \Lambda^{p,q} \to \Lambda^{p,q+1}$ as follows: Let $\phi \in \Lambda^{p,q}$ and let $X_1, \ldots, X_p$ (resp., $Y_1, \ldots, Y_{q+1}$) be sections of $T_{1,0}$ (resp., $T_{0,1}$). Then

$$<\bar{\partial}_b\phi;(X_1\wedge\ldots\wedge X_p)\otimes(Y_1\wedge\ldots\wedge Y_{q+1})>$$

$$=(q+1)^{-1}\sum_{j=1}^{q+1}(-1)^{j+1}Y_j<\phi;(X_1\wedge\ldots\wedge X_p)\otimes(Y_1\wedge\ldots\hat{Y}_j\ldots\wedge Y_{q+1})>$$

$$+(q+1)^{-1}\sum_{i<j}(-1)^{i+j}<\phi;(X_1\wedge\ldots\wedge X_p)\otimes([Y_i,Y_j]\wedge Y_1\wedge\ldots\hat{Y}_i\ldots\hat{Y}_j\ldots\wedge Y_{q+1})>.$$

The ^ symbol over a section means as usual that it is deleted from the expression. One can show that

i) $\bar{\partial}_b^2=0$,

ii) $\bar{\partial}_b(\phi\wedge\psi)=(\bar{\partial}_b\phi)\wedge\psi+(-1)^p\phi\wedge\bar{\partial}_b\psi$ for $\phi\in\Lambda^{0,p}$,

iii) $<\bar{\partial}_b f, Y>=Yf$ for $f\in\Lambda^{0,0}$ and Y a section of $T_{0,1}$.

See [6] for details.

The Heisenberg group, H, is a Lie group with a natural CR structure. The manifold is $\mathbf{C}^n\times\mathbf{R}$. Let (z,t), $(z',t')\in\mathbf{C}^n\times\mathbf{R}=H$. The group law is defined by

$$(z,t)\cdot(z',t')=\big(z+z',t+t'+2\operatorname{Im}(z\cdot z')\big)$$

where $z\cdot z'=\sum_{j=1}^{n}z_j\bar{z}'_j$. The identity element is $(0,0)$ and $(z,t)^{-1}=(-z,-t)$.

Sometimes we will set $u=(z,t)$.

For $j=1,\ldots,n$ if we set $z_j=x_j+iy_j$, the mapping

$$(z,t)\to(x_1,\ldots,x_n,y_1,\ldots,y_n,t)$$

defines a C^∞ coordinate system on H. The left invariant vector fields (i.e., the elements of the Lie algebra) are $\mathbf{R}$-linear combinations of

$$X_j=\frac{\partial}{\partial x_j}+2y_j\frac{\partial}{\partial t},\ Y_j=\frac{\partial}{\partial y_j}-2x_j\frac{\partial}{\partial t},\ T=\frac{\partial}{\partial t}. \tag{2}$$

They satisfy the following commutation relations:

$$[X_j,T]=[Y_j,T]=[X_j,X_k]=[Y_j,Y_k]=0, \tag{3}$$

$$[X_j,Y_k]=-4\delta_{jk}T. \tag{4}$$

Let $\mathbf{C}T$ be the complexified tangent bundle of H. Define

$$Z_j=\frac{1}{2}(X_j-iY_j)=\frac{\partial}{\partial z_j}+i\bar{z}_j\frac{\partial}{\partial t}. \tag{5}$$

Then Z_j, $\bar{Z}_j$, and T form a basis (over $\mathbf{R}$) of the space of left invariant complex tangent vector fields. In particular, they form a global frame of $\mathbf{C}T$. From (3), (4) and (5) we easily see that

$$(6) \qquad [Z_j, Z_k] = [\bar{Z}_j, \bar{Z}_k] = [Z_j, T] = [\bar{Z}_j, T] = 0$$
$$[Z_j, \bar{Z}_k] = -2i\delta_{jk}T .$$

Let $T_{1,0}$ (resp., $T_{0,1}$) be the subbundle of $\mathbf{C}T$ spanned by the Z_j's (resp., $\bar{Z}_j$'s). Then

$$\bar{T}_{1,0} = T_{0,1}$$
$$T_{1,0} \cap T_{0,1} = \{0\}$$
$$\operatorname{codim}(T_{1,0} \otimes T_{0,1}) = 1 .$$

Also, if V_1, V_2 are sections of $T^{1,0}$ we can write $V_i = \sum_{j=1}^{n} f_{ij}Z_j, i = 1, 2$ where the f_{ij} are C^∞ functions on H. Then by (6),

$$[V_1, V_2] = \sum_{k=1}^{n}\left(\sum_{j=1}^{n}(f_{1j}Z_jf_{2k} - f_{2j}Z_jf_{1k})\right)Z_k .$$

So, the splitting of $\mathbf{C}T$ defines a CR structure on H.

Impose the left invariant Hermitian metric on $\mathbf{C}T$ which makes the Z's, $\bar{Z}$'s and T an orthonormal frame. Let ω^j and τ be dual to Z_j and T, respectively. Then ω^j, $\bar{\omega}^j$ and τ form an orthonormal frame for $\mathbf{C}T^*$. The volume element on H is

$$(7) \qquad du = 2^n dx_1 \wedge \dots \wedge dx_n \wedge dy_1 \wedge \dots \wedge dy_n \wedge dt .$$

Since H is nilpotent and, hence, unimodular, the volume element is both left and right invariant. One can also verify this directly.

Let $J = (j_1, \dots, j_q)$ be a multi-index with $1 \leqslant j_i \leqslant n, i = 1, \dots, q$. Define $|J| = q$ and $\bar{\omega}^J = \bar{\omega}^{j_1} \wedge \dots \wedge \bar{\omega}^{j_q}$. If $\phi \in \Lambda^{0,q}(H)$ we may write $\phi = \sum_{|J|=q} \phi_J\bar{\omega}^J$ where ϕ_J is a C^∞ function from H to $\mathbf{C}$. Let

$$\bar{\omega}^j \lrcorner \bar{\omega}^J = (-1)^k \bar{\omega}^{j_1} \wedge \dots \wedge \widehat{\bar{\omega}^{j_k}} \wedge \dots \wedge \bar{\omega}^{j_q} \quad \text{if} \quad j = j_k \quad \text{and} \quad \bar{\omega}^j \lrcorner \bar{\omega}^J = 0$$

otherwise. Folland and Stein prove that for $\phi \in \Lambda^{0,q}$

i) $\bar{\partial}_b\phi = \sum_{|J|=q} \sum_{j=1}^{n} \bar{Z}_j\phi_J\bar{\omega}^j \wedge \bar{\omega}^J ,$

$$\text{(8)} \quad \text{ii)} \quad \bar{\partial}_b^* \phi = - \sum_{|J|=q} \sum_{j=1}^{n} Z_j \phi_J \bar{\omega}^j \lrcorner \bar{\omega}^J ,$$

$$\text{iii)} \quad \Box_b \phi = \sum_{|J|=q} \left(-\frac{1}{2} \sum_{j=1}^{n} (Z_j \bar{Z}_j + \bar{Z}_j Z_j) + i(n-2q)T \right) \phi_J \bar{\omega}^J .$$

Define the function

$$\Phi_\alpha(z, t) = (|z|^2 - it)^{-\frac{n+\alpha}{2}} (|z|^2 + it)^{-\frac{n-\alpha}{2}} .$$

Let $\phi \in \Lambda_c^{0,q}$, $q \neq 0, n$. For an appropriate constant, c_q, define

$$\text{(9)} \qquad K_q \phi(v) = c_q \sum_{|J|=q} \left(\int_H \phi_J(u) \Phi_{n-2q}(u^{-1}v) du \right) \bar{\omega}^J .$$

Folland and Stein prove that for the appropriate c_q

THEOREM 1. *Let* $\phi \in \Lambda_c^{0,q}$, $q \neq 0, n$. *Then* $\Box_b K_q \phi = K_q \Box_b \phi = \phi$.

In [4] we prove a stronger version of the following Hodge decomposition theorem.

THEOREM 2. *Let* $\phi \in \Lambda_c^{0,q}$, $q \neq 0, n$. *Then*

i) $H\phi = 0$ *where* H *is the orthogonal projection onto the kernel of* $\Box_b$.

ii) $\phi = \bar{\partial}_b \bar{\partial}_b^* K_q \phi + \bar{\partial}_b^* \bar{\partial}_b K_q \phi$.

We also prove

THEOREM 3. *If* $\phi \in \Lambda_c^{0,q}$, $q \neq 0, n$ *and if* $\bar{\partial}_b \phi = 0$ *then* $\psi = \bar{\partial}_b^* K_q \phi$ *satisfies* $\bar{\partial}_b \psi = \phi$.

These two theorems are special cases of theorems 6 and 7 proven in section 4.

3. DIFFERENTIAL COMPLEXES ON STRATIFIED GROUPS

We study a class of nilpotent Lie groups which we describe in terms of their Lie algebras. A graded Lie algebra, $\mathfrak{n}$, is a finite dimensional nilpotent algebra which has a direct sum decomposition, $\mathfrak{n} = \bigoplus_{i=1}^{r} \mathfrak{n}_i$ where the $\mathfrak{n}_i$ satisfy

i) $[\mathfrak{n}_i, \mathfrak{n}_j] \subseteq \mathfrak{n}_{i+j}$ if $i + j \leqslant r$,

ii) $[\mathfrak{n}_i, \mathfrak{n}_j] = 0$ if $i + j > r$.