GEOMETRIC PROOF OF BIEBERBACH'S THEOREMS ON CRYSTALLOGRAPHIC GROUPS

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A GEOMETRIC PROOF OF BIEBERBACH'S THEOREMS ON CRYSTALLOGRAPHIC GROUPS

by Peter Buser

Pour Ariane et Georges

1. Introduction

In 1910 Bieberbach proved two celebrated theorems in response to Hilbert's 18th problem.

Theorem I. *Every discrete group of isometries acting on the n-dimensional euclidean space* $\mathbf{R}^n$ *with compact fundamental domain contains* n *linearly independent translations.*

Groups which satisfy the hypothesis of Theorem I are called n-dimensional crystallographic groups.

Theorem II. *For each fixed* n *there are only finitely many isomorphism classes of n-dimensional crystallographic groups.*

Bieberbach's original proof of Theorem I is based on Minkowski's Theorem on simultaneous rational approximation and is difficult to read. Shortly after it came out, Frobenius gave a more accessible proof which is based on an argument using the commutativity of unitary matrices. Frobenius's method has, in one form or another, become standard in the contemporary literature.

In this note we present a completely different approach to Theorem I which has its origins in Gromov's work on almost flat manifolds [5]. The new idea is to start with those rigid motions which have a very small rotation part (cf. § 2 for notation), and then proceed to show that, in fact, these motions are pure translations. The simplification which results from this approach is striking.

We also give a new proof of Theorem II which does not run via the usual algebraic characterization of a crystallographic group. Instead we shall use a method which is more in the spirit of Minkowski's geometry of numbers, from where Bieberbach's original arguments departed.

Since the material is standard, the exposition will be condensed. Yet some efforts have been made not to frustrate the reader by omitting details.

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2. Rigid motions

In this section we fix the notation and collect the necessary (and hopefully sufficient) rudiments from Linear Algebra.

We consider $\mathbf{R}^n$ as an euclidean vector space with the standard inner product. We use $|x|$ to denote the length of a vector $x \in \mathbf{R}^n$, and $\measuredangle (x, y) \in [0, \pi]$ to denote the angle between two vectors. A rigid motion α (isometry of $\mathbf{R}^n$) will be expressed in the form

$$x \mapsto \alpha x = Ax + a \qquad (x \in \mathbf{R}^n)$$

where $A = \operatorname{rot} \alpha \in O(n)$ is an orthogonal map, called the *rotation part* of α, and $\alpha = \operatorname{trans} \alpha \in \mathbf{R}^n$ is a vector, called the *translation part*.

2.1. *The commutator* $[\alpha, \beta]$ of two rigid motions $x \mapsto \alpha x = Ax + a$ and $x \mapsto \beta x = Bx + b$ is defined as $[\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1}$. The following formulae are easily checked:

$$\operatorname{rot} [\alpha, \beta] = [A, B]$$

$$\operatorname{trans} [\alpha, \beta] = (A-id)b + (id-[A, B])b + A(id-B)A^{-1}a \,.$$

2.2. *Rotations.* For $A \in O(n)$ we define

$$m(A) = \max\left\{ \, |Ax - x| / |x| \, \Big| \, x \in \mathbf{R}^n \backslash \{0\} \right\}.$$

Note that $|Ax - x| \leqslant m(A) |x|$ for $x \in \mathbf{R}^n$. *The set*

(i) $$E_A = \left\{ \, x \in \mathbf{R}^n \, \Big| \, |Ax - x| = m(A) |x| \, \right\}$$

is a non trivial A-invariant subspace. This is immediately checked except perhaps for the part "$x, y \in E_A$ implies $x \pm y \in E_A$". This part follows from the equation

$$2m^2(A)(|x|^2+|y|^2) = 2(|Ax-x|^2+|Ay-y|^2) = |A(x+y) - (x+y)|^2$$
$$+ |A(x-y) - (x-y)|^2 \leqslant m^2(A)(|x+y|^2+|x-y|^2) = 2m^2(A)(|x|^2+|y|^2)$$

Since A is an orthogonal map, the orthogonal complement $E_A^\perp$ of E_A is also an A-invariant linear subspace of $\mathbf{R}^n$. We define

(ii) $$m^\perp(A) = \max\{\,|Ax - x|/|x| \mid x \in E_A^\perp \setminus \{0\}\,\}$$

if $E_A^\perp \neq \{0\}$ and set $m^\perp(A) = 0$ if $E_A^\perp = \{0\}$. It follows that

(iii) $$m^\perp(A) < m(A) \text{ if } A \neq id\,.$$

We let $x = x^E + x^\perp$, $x^E \in E_A$, $x^\perp \in E_A^\perp$ be the orthogonal decomposition of a vector x with respect to E_A and $E_A^\perp$. The simple observation

(iv) $$|Ax^E - x^E| = m(A)\,|x^E|\,, \quad |Ax^\perp - x^\perp| \leqslant m^\perp(A)\,|x^\perp|$$

together with (iii), will play a crucial role in the proof of Theorem I.

2.3. *Commutator estimate. For* $A, B \in O(n)$ *we have*

$$m([A, B]) \leqslant 2m(A)\,m(B)\,.$$

Proof. Verify the identity

$$[A, B] - id = \big((A-id)(B-id) - (B-id)(A-id)\big)A^{-1}B^{-1}$$

From $|A^{-1}B^{-1}x| = |x|$ it then follows that

$$|[A, B]\,x - x| \leqslant m(A)\,m(B)\,|x| + m(B)\,m(A)\,|x|$$

for all $x \in \mathbf{R}^n$.

2.4. *Crystallographic groups.* Discreteness and compactness of the fundamental domain will be used as follows:

A group G of rigid motions in $\mathbf{R}^n$ is called crystallographic if

(i) for all $t > 0$ only finitely many $\alpha \in G$ have $|a| \leqslant t$,

(ii) there is some constant d such that for each $x \in \mathbf{R}^n$ there is an element $\alpha \in G$ satisfying $|a - x| \leqslant d$.

3. Proof of Theorem I

Now let G be an n-dimensional crystallographic group.

3.1. Lemma A ("Mini Bieberbach"). *For each unit vector* $u \in \mathbf{R}^n$ *and for all* $\varepsilon, \delta > 0$ *there exists* $\beta \in G$ *satisfying*

$$b \neq 0\,, \qquad \sphericalangle\,(u, b) \leqslant \delta\,, \qquad m(B) \leqslant \varepsilon\,.$$

Proof. By 2.4 (ii) there exists an element $\beta_k \in G$ satisfying $|\, b_k - k\,u \,| \leqslant d$, for each $k = 1, 2 \ldots$. The sequence $\beta_1, \beta_2, \ldots$ satisfies

$$|\, b_k \,| \to \infty\,, \qquad \sphericalangle\,(u, b_k) \to 0 \qquad (k \to \infty)\,.$$

Since $O(n)$ is compact, we find a subsequence such that the rotation parts B_k also converge. Consequently there exist $i < j$ such that

$$m(B_j B_i^{-1}) \leqslant \varepsilon\,, \qquad \sphericalangle\,(u, b_j) \leqslant \delta/2\,, \qquad |\, b_i \,| \leqslant \frac{\delta}{4} \,|\, b_j \,|\,.$$

The motion $x \mapsto \beta_j \beta_i^{-1} x = B_j B_i^{-1} x + b_j - B_j B_i^{-1} b_i$ has now all the required properties.

3.2. Lemma B. *If* $\alpha \in G$ *satisfies* $m(A) \leqslant \frac{1}{2}$, *then* α *is a pure translation.*

Proof. If G contains elements α satisfying $0 < m(A) \leqslant \frac{1}{2}$, we consider the one for which $|\, a \,|$ is a *minimum* (2.4 (i)). Lemma A (applied to an arbitrary unit vector $u \in E_A$) provides elements $\beta \in G$ satisfying

$$(*) \qquad b \neq 0\,, \quad |\, b^{\perp} \,| \leqslant |\, b^{E} \,|\,, \quad m(B) \leqslant \frac{1}{8}\big(m(A) - m^{\perp}(A)\big)$$

(c.f. 2.2. (iii)). Among these we again consider the one for which $|\, b \,|$ is a minimum ($\neq 0$!). Observe that $|\, b \,| \geqslant |\, a \,|$ if β is not a translation by the choice of α.

The commutator $\tilde{\beta} = [\alpha, \beta]$ satisfies

$$m(\tilde{B}) = m([A, B]) \leqslant 2m(A)\, m(B) \leqslant m(B)$$

(2.3), and we have by 2.1

$$\tilde{b} = (A - id) b^{E} + (A - id) b^{\perp} + r\,,$$
$$r = (id - \tilde{B}) b + A(id - B) A^{-1} a\,.$$

If β is a translation, then $B = id = \tilde{B}$ and therefore $r = 0$.

If β is not a translation, then $|\, a \,| \leqslant |\, b \,|$ (by the choice of α) and therefore $|\, r \,| \leqslant \big(m(\tilde{B}) + m(B)\big)\,|\, b \,| \leqslant 2m(B)\,|\, b \,| < 4m(B)\,|\, b^{E} \,|$. Hence, in either case,

$$| r | < \frac{1}{2}\left(m(A) - m^{\perp}(A)\right) | b^E | .$$

Together with 2.2 (iv) we obtain

$$| \tilde{b}^{\perp} | < \frac{1}{2}\left(m(A) + m^{\perp}(A)\right) | b^E | < | \tilde{b}^E | .$$

We find that $\tilde{\beta}$ also satisfies (*), but with $| \tilde{b} | \leqslant m(A) | b | - r < | b |$, a contradiction.

3.3. *End of proof.* Lemma A provides elements in G with n linearly independent translation parts whose rotation parts are smaller than $\frac{1}{2}$. By Lemma B these elements are pure translations.

4. Lattices

In this paragraph we collect the rudiments from lattice point theory which are necessary for the proof of Theorem II. A lattice L is a crystallographic group which consists only of translations. The elements of L (lattice points) will be identified with vectors in $\mathbf{R}^n$. By abuse of notation, we shall write $\omega = w = \operatorname{trans}\omega$ for $\omega \in L$. It is well known that L is isomorphic to $\mathbf{Z}^n$ but this fact will *not* be used in our proof of Theorem II. Notice, however that L is abelian and that the minimal distance of lattice points equals the length of the smallest non-zero element in L.

4.1. Lemma. *Let L be a lattice in $\mathbf{R}^n$ whose elements have pairwise distances $\geqslant 1$, and let $N(\rho)$ denote the number of lattice points in L whose distance from the origin is $\leqslant \rho$ $(\rho > 0)$. Then*

$$N(\rho) \leqslant (2\rho + 1)^n .$$

Proof. The open balls of radius $\frac{1}{2}$ around the $N(\rho)$ lattice points are pairwise disjoint and all contained in a ball of radius $\rho + \frac{1}{2}$. Comparing the volumes we find $N(\rho)\left(\frac{1}{2}\right)^n \leqslant \left(\rho + \frac{1}{2}\right)^n$.

4.2. LEMMA. *Let L be a lattice in $\mathbf{R}^n$ whose elements have pairwise distances $\geqslant 1$ and consider a linear subspace E of $\mathbf{R}^n$ which is spanned by k vectors $w_1, \ldots, w_k \in L$. If a lattice point $w \in L$ is not contained in E, then its $E^{\perp}$-component $w^{\perp}$ has length*

$$| w^{\perp} | \geqslant (3 + |w_1| + \ldots + |w_k|)^{-n}.$$

Proof. Let N be the integer part of $(3+|w_1|+\ldots+|w_k|)^n$. If $0 < | w^{\perp} | \leqslant 1/N$, then $0, w, 2w, \ldots, Nw$ have distance $\leqslant 1$ from E. Adding suitable integer linear combinations of $w_1, \ldots, w_k$ to each of these vectors we obtain $N + 1$ new pairwise different lattice points whose $E^{\perp}$ components have not changed but whose E components are $\leqslant \frac{1}{2}(|w_1|+\ldots+|w_k|)$. These $N + 1$ lattice points have distance $\leqslant 1 + \frac{1}{2}(|w_1|+\ldots+|w_k|)$ from the origin, a contradiction to Lemma 4.1.

5. PROOF OF THEOREM II

For an n-dimensional crystallographic group G we let $L(G)$ be the subgroup consisting of all pure translations in G. By Theorem I, $L(G)$ is a lattice in $\mathbf{R}^n$. The standard observation which is "responsible" for Theorem II is

5.1. LEMMA. *If $\alpha \in G$ and if $w \in L(G)$, then $A(w) \in L(G)$, $(A = \text{rot}\, \alpha)$.*

Proof. Recall that $w = \text{trans}\, \omega$, $\omega \in L(G)$. Now $\alpha\omega\alpha^{-1} \in G$ is a translation with translation vector $A(w)$. Hence $A(w) \in L(G)$.

5.2. *Definition.* A crystallographic group is called *normal* if

(i) the vectors in $L(G)$ have pairwise distances $\geqslant 1$

(ii) $L(G)$ contains n linearly independent *unit* vectors.

We do not ask that the vectors in (ii) generate the entire lattice $L(G)$.

Our idea is to count the normal groups. This will suffice due to the following.

5.3. PROPOSITION. *Each crystallographic group G is isomorphic to a normal crystallographic group.*

Proof. By scaling we may assume that the shortest non zero vector in $L(G)$ is a unit vector. Now assume by induction that $L(G)$ satisfies 5.2 (i) and contains $k < n$ unit vectors $w_1, ..., w_k$ which span a k-dimensional linear subspace E of $\mathbf{R}^n$. It remains to find a group G' isomorphic to G such that $L(G)$ contains $k+1$ linearly independent unit vectors and also satisfies 5.2 (i).

If for some $\alpha \in G$ and for some $w_i (i \leqslant k)$ the vector $A(w_i)$ is not contained in E, then by Lemma 5.1 $A(w_i) \in L(G)$ is already the $(k+1)$-st vector and we are done.

If on the other hand all rotation parts of G leave E—and consequently $E^\perp$—invariant, then the affine transformations Φ_μ given by

$$\Phi_\mu(x^E + x^\perp) = x^E + \mu x^\perp$$

$(\mu > 0)$ commute with the rotation parts of G. Therefore, the affine conjugate (and henceforth isomorphic) groups $G_\mu = \Phi_\mu \, G \, \Phi_\mu^{-1}$ *also act by rigid motions.* Since $L(G_\mu) = \Phi_\mu(L(G))$, Lemma 4.2 implies that G_μ violates 5.2 (i) if $\mu > 0$ is very small. Hence there exists a *minimal* $\mu' > 0$ such that $G_{\mu'}$ satisfies 5.2 (i). Since the affine transformations Φ_μ act trivially on E, the shortest vector in $L(G_{\mu'}) \backslash E$ must be a unit vector and $w_1, ..., w_k \in L(G_{\mu'})$. Now $G_{\mu'}$ has the required properties and Proposition 5.3 is proved.

5.4. *The proof* of Theorem II now proceeds in two steps.

Step 1. Each normal crystallographic group G is uniquely characterized by a group table ((ii) below).

Proof. Fix n linearly independent unit vectors $w_1, ... w_n \in L(G)$ and consider the sublattice

$$L = \{m_1 w_1 + ... + m_n w_n \mid m_1, ..., m_n \in \mathbf{Z}\} .$$

L is a subgroup of G. In each right coset modulo L of G we select a representative ω whose translation part w has length

(i) $$|w| \leqslant \frac{1}{2}(|w_1| + ... + |w_n|) = \frac{n}{2} .$$

Since G is discrete (2.4. (i)), there are only finitely many such representatives, say $\omega_{n+1}, ..., \omega_N$. Every $\alpha \in G$ can now be expressed in a unique way in the form

$$\alpha = (m_1 w_1 + ... + m_n w_n)\omega_\nu$$

where $n + 1 \leqslant \nu \leqslant N$. Since our L is isomorphic to $\mathbf{Z}^n$, G is uniquely determined (up to isomorphism) by the integers m_{ijk}, $\nu(j, k)$ and N which occur in the table

$$\text{(ii)} \qquad \omega_j \omega_k = (m_{1jk} w_1 + \ldots + m_{njk} w_n)\, \omega_{\nu(j,k)}, \qquad j, k = 1, \ldots, N.$$

(For $i = 1, \ldots, n$, ω_i is the translation by w_i).

Clearly, the proof of Theorem II will be completed by

Step 2. The absolute values of m_{ijk}, $\nu(j, k)$ *and* N *in* (ii) *have an upper bound which depends only on the dimension* n *(see* (iii) *and* (iv) *below).*

Proof. The euclidean motions $\omega_{\nu(j,k)}$, ω_j and ω_k in (ii) have translation parts of length $\leqslant \frac{n}{2}$ (c.f. (i)). Consequently the translation $m_{1jk} w_1 + \ldots + m_{njk} w_n = \omega_j \omega_k \omega_{\nu(j,k)}^{-1}$ has length $\leqslant \frac{3n}{2}$. In particular,

$$| m_{ijk}\, w_i^{\perp} | \leqslant \frac{3n}{2}, \quad i = 1, \ldots, n$$

where $w_i^{\perp}$ is the component of w_i perpendicular to the hyperplane E spanned by $w_1, \ldots, w_{i-1}, w_{i+1}, \ldots, w_n$. By Lemma 4.2 we have $| w_i^{\perp} | \geqslant (n+2)^{-n}$. Hence

$$\text{(iii)} \qquad | m_{ijk} | \leqslant \frac{3n}{2} (n+2)^n .$$

Now let us estimate N. The linear transformation $A = \operatorname{rot} \alpha$, $\alpha \in G$ is uniquely determined by its images $A(w_i)$, $i = 1, \ldots, n$. By Lemma 5.1 each of these images is a unit vector of $L(G)$ and, by Lemma 4.1, one out of at most 3^n candidates. It follows that at most $(3^n)^n$ different rotation parts occur in G.

If two elements ω_ρ and ω_σ among $\omega_{n+1}, \ldots, \omega_N$ have the same rotation part, then $\omega_\rho \omega_\sigma^{-1}$ is a vector of length $\leqslant \frac{n}{2} + \frac{n}{2}$ (c.f. (i)) and, again by Lemma 4.1, one out of at most $(2n+1)^n$ candidates. Hence

$$\text{(iv)} \qquad N \leqslant n + (3^n)^n \cdot (2n+1)^n .$$

Since $\nu(i, j) \leqslant N$, this concludes the proof of Theorem II.

5.5. *Remark.* From the preceding proof we can derive the upper bound $\exp \exp 4n^2$ for the number of isomorphism classes of n-dimensional crystallographic groups. The correct numbers for $n = 1, 2, 3, 4$ are respectively 2, 17, 219, 4783 [4].

REFERENCES

The original articles of Bieberbach and Frobenius are

[1] BIEBERBACH, L. Über die Bewegungsgruppen der Euklidischen Räume, I: *Math. Ann. 70* (1911), 297-336; II: *Math. Ann. 72* (1912), 400-412.

[2] FROBENIUS, C. Über die unzerlegbaren diskreten Bewegungsgruppen. *Sitzungsber. Akad. Wiss. Berlin 29* (1911), 654-665.

A simplified version of Frobenius'proof using minor amounts of Lie group theory is in

[3] AUSLANDER, L. An account of the theory of crystallographic groups. *Proc. Amer. Math. Soc. 16* (1956), 1230-1236.

For historical remarks we refer to

[4] BROWN, H., R. BÜLOW, J. NEUBÜSER, H. WONDRATSCHEK and H. ZASSENHAUS. *Crystallographic Groups of Four Dimensional Space.* New York, Wiley, 1978.

[5] BUSER, P. and H. KARCHER. The Bieberbach Case in Gromov's Almost Flat Manifold Theorem. In *Global Differential Geometry and Global Analysis, Proceedings, Berlin 1979, Lect. Notes in Mathematics, 838*, Berlin, Springer Verlag, 1981.

[6] MILNOR, J. Hilbert's Problem 18: On Crystallographic Groups, Fundamental Domains, and on Sphere Packing. *Proceedings of Symposia in Pure Mathematics, 28*, Amer. Math. Soc., Providence, 1976.

A proof of Bieberbach's third theorem that two crystallographic groups are isomorphic if and only if they are conjugate by an affine transformation, may be found on p. 375 in

[7] RINOW, W. *Die innere Geometrie der metrischen Räume.* Berlin, Springer Verlag, 1961.

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