3. Proof of Theorem I

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Since A is an orthogonal map, the orthogonal complement $E_A^\perp$ of E_A is also an A-invariant linear subspace of $\mathbf{R}^n$. We define

(ii) $$m^\perp(A) = \max \{ \mid Ax - x \mid / \mid x \mid \; \mid \; x \in E_A^\perp \setminus \{0\} \}$$

if $E_A^\perp \neq \{0\}$ and set $m^\perp(A) = 0$ if $E_A^\perp = \{0\}$. It follows that

(iii) $$m^\perp(A) < m(A) \text{ if } A \neq id .$$

We let $x = x^E + x^\perp$, $x^E \in E_A$, $x^\perp \in E_A^\perp$ be the orthogonal decomposition of a vector x with respect to E_A and $E_A^\perp$. The simple observation

(iv) $$\mid Ax^E - x^E \mid = m(A) \mid x^E \mid , \quad \mid Ax^\perp - x^\perp \mid \leqslant m^\perp(A) \mid x^\perp \mid$$

together with (iii), will play a crucial role in the proof of Theorem I.

2.3. *Commutator estimate. For* $A, B \in O(n)$ *we have*

$$m([A, B]) \leqslant 2m(A)\, m(B) .$$

Proof. Verify the identity

$$[A, B] - id = \big((A-id)(B-id) - (B-id)(A-id)\big)A^{-1}B^{-1}$$

From $\mid A^{-1}B^{-1}x \mid = \mid x \mid$ it then follows that

$$\mid [A, B]\, x - x \mid \leqslant m(A)\, m(B) \mid x \mid + m(B)\, m(A) \mid x \mid$$

for all $x \in \mathbf{R}^n$.

2.4. *Crystallographic groups.* Discreteness and compactness of the fundamental domain will be used as follows:

A group G of rigid motions in $\mathbf{R}^n$ is called crystallographic if

(i) for all $t > 0$ only finitely many $\alpha \in G$ have $\mid a \mid \leqslant t$,

(ii) there is some constant d such that for each $x \in \mathbf{R}^n$ there is an element $\alpha \in G$ satisfying $\mid a - x \mid \leqslant d$.

3. Proof of Theorem I

Now let G be an n-dimensional crystallographic group.

3.1. Lemma A ("Mini Bieberbach"). *For each unit vector* $u \in \mathbf{R}^n$ *and for all* $\varepsilon, \delta > 0$ *there exists* $\beta \in G$ *satisfying*

$$b \neq 0\,, \qquad \sphericalangle\,(u, b) \leqslant \delta\,, \qquad m(B) \leqslant \varepsilon\,.$$

Proof. By 2.4 (ii) there exists an element $\beta_k \in G$ satisfying $| b_k - k\,u | \leqslant d$, for each $k = 1, 2...$. The sequence $\beta_1, \beta_2, ...$ satisfies

$$| b_k | \to \infty\,, \qquad \sphericalangle\,(u, b_k) \to 0 \qquad (k \to \infty)\,.$$

Since $O(n)$ is compact, we find a subsequence such that the rotation parts B_k also converge. Consequently there exist $i < j$ such that

$$m(B_j B_i^{-1}) \leqslant \varepsilon\,, \qquad \sphericalangle\,(u, b_j) \leqslant \delta/2\,, \qquad | b_i | \leqslant \frac{\delta}{4} | b_j |\,.$$

The motion $x \mapsto \beta_j \beta_i^{-1} x = B_j B_i^{-1} x + b_j - B_j B_i^{-1} b_i$ has now all the required properties.

3.2. Lemma B. *If* $\alpha \in G$ *satisfies* $m(A) \leqslant \frac{1}{2}$, *then* α *is a pure translation.*

Proof. If G contains elements α satisfying $0 < m(A) \leqslant \frac{1}{2}$, we consider the one for which $| a |$ is a *minimum* (2.4 (i)). Lemma A (applied to an arbitrary unit vector $u \in E_A$) provides elements $\beta \in G$ satisfying

$$(*) \qquad b \neq 0\,, \quad | b^{\perp} | \leqslant | b^E |\,, \quad m(B) \leqslant \frac{1}{8}\left(m(A) - m^{\perp}(A)\right)$$

(c.f. 2.2. (iii)). Among these we again consider the one for which $| b |$ is a minimum ($\neq 0$!). Observe that $| b | \geqslant | a |$ if β is not a translation by the choice of α.

The commutator $\tilde{\beta} = [\alpha, \beta]$ satisfies

$$m(\tilde{B}) = m([A, B]) \leqslant 2m(A)\, m(B) \leqslant m(B)$$

(2.3), and we have by 2.1

$$\tilde{b} = (A - id) b^E + (A - id) b^{\perp} + r\,,$$
$$r = (id - \tilde{B}) b + A(id - B) A^{-1} a\,.$$

If β is a translation, then $B = id = \tilde{B}$ and therefore $r = 0$.

If β is not a translation, then $| a | \leqslant | b |$ (by the choice of α) and therefore $| r | \leqslant \left(m(\tilde{B}) + m(B)\right) | b | \leqslant 2m(B) | b | < 4m(B) | b^E |$. Hence, in either case,

$$|r| < \frac{1}{2}(m(A) - m^{\perp}(A)) \, |b^E| \, .$$

Together with 2.2 (iv) we obtain

$$|\tilde{b}^{\perp}| < \frac{1}{2}(m(A) + m^{\perp}(A)) \, |b^E| < |\tilde{b}^E| \, .$$

We find that $\tilde{\beta}$ also satisfies (*), but with $|\tilde{b}| \leqslant m(A)\,|b| - r < |b|$, a contradiction.

3.3. *End of proof.* Lemma A provides elements in G with n linearly independent translation parts whose rotation parts are smaller than $\frac{1}{2}$. By Lemma B these elements are pure translations.

4. Lattices

In this paragraph we collect the rudiments from lattice point theory which are necessary for the proof of Theorem II. A lattice L is a crystallographic group which consists only of translations. The elements of L (lattice points) will be identified with vectors in $\mathbf{R}^n$. By abuse of notation, we shall write $\omega = w = \operatorname{trans} \omega$ for $\omega \in L$. It is well known that L is isomorphic to $\mathbf{Z}^n$ but this fact will *not* be used in our proof of Theorem II. Notice, however that L is abelian and that the minimal distance of lattice points equals the length of the smallest non-zero element in L.

4.1. Lemma. *Let L be a lattice in $\mathbf{R}^n$ whose elements have pairwise distances $\geqslant 1$, and let $N(\rho)$ denote the number of lattice points in L whose distance from the origin is $\leqslant \rho$ $(\rho > 0)$. Then*

$$N(\rho) \leqslant (2\rho + 1)^n \, .$$

Proof. The open balls of radius $\frac{1}{2}$ around the $N(\rho)$ lattice points are pairwise disjoint and all contained in a ball of radius $\rho + \frac{1}{2}$. Comparing the volumes we find $N(\rho)\left(\frac{1}{2}\right)^n \leqslant \left(\rho + \frac{1}{2}\right)^n$.