5. Proof of Theorem II

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4.2. Lemma. *Let* L *be a lattice in* $\mathbf{R}^n$ *whose elements have pairwise distances* $\geqslant 1$ *and consider a linear subspace* E *of* $\mathbf{R}^n$ *which is spanned by* k *vectors* $w_1, \ldots, w_k \in L$. *If a lattice point* $w \in L$ *is not contained in* E, *then its* $E^\perp$*-component* $w^\perp$ *has length*

$$|w^\perp| \geqslant (3+|w_1|+\ldots+|w_k|)^{-n}.$$

Proof. Let N be the integer part of $(3+|w_1|+\ldots+|w_k|)^n$. If $0 < |w^\perp| \leqslant 1/N$, then $0, w, 2w, \ldots, Nw$ have distance $\leqslant 1$ from E. Adding suitable integer linear combinations of $w_1, \ldots, w_k$ to each of these vectors we obtain $N+1$ new pairwise different lattice points whose $E^\perp$ components have not changed but whose E components are $\leqslant \frac{1}{2}(|w_1|+\ldots+|w_k|)$. These $N+1$ lattice points have distance $\leqslant 1 + \frac{1}{2}(|w_1|+\ldots+|w_k|)$ from the origin, a contradiction to Lemma 4.1.

5. Proof of Theorem II

For an n-dimensional crystallographic group G we let $L(G)$ be the subgroup consisting of all pure translations in G. By Theorem I, $L(G)$ is a lattice in $\mathbf{R}^n$. The standard observation which is "responsible" for Theorem II is

5.1. Lemma. *If* $\alpha \in G$ *and if* $w \in L(G)$, *then* $A(w) \in L(G)$, $(A = \text{rot}\,\alpha)$.

Proof. Recall that $w = \text{trans}\,\omega$, $\omega \in L(G)$. Now $\alpha\omega\alpha^{-1} \in G$ is a translation with translation vector $A(w)$. Hence $A(w) \in L(G)$.

5.2. *Definition.* A crystallographic group is called *normal* if

(i) the vectors in $L(G)$ have pairwise distances $\geqslant 1$

(ii) $L(G)$ contains n linearly independent *unit* vectors.

We do not ask that the vectors in (ii) generate the entire lattice $L(G)$.

Our idea is to count the normal groups. This will suffice due to the following.

5.3. Proposition. *Each crystallographic group* G *is isomorphic to a normal crystallographic group.*

Proof. By scaling we may assume that the shortest non zero vector in $L(G)$ is a unit vector. Now assume by induction that $L(G)$ satisfies 5.2 (i) and contains $k < n$ unit vectors $w_1, ..., w_k$ which span a k-dimensional linear subspace E of $\mathbf{R}^n$. It remains to find a group G' isomorphic to G such that $L(G)$ contains $k+1$ linearly independent unit vectors and also satisfies 5.2 (i).

If for some $\alpha \in G$ and for some $w_i (i \leqslant k)$ the vector $A(w_i)$ is not contained in E, then by Lemma 5.1 $A(w_i) \in L(G)$ is already the $(k+1)$-st vector and we are done.

If on the other hand all rotation parts of G leave E—and consequently $E^\perp$—invariant, then the affine transformations Φ_μ given by

$$\Phi_\mu(x^E + x^\perp) = x^E + \mu x^\perp$$

$(\mu > 0)$ commute with the rotation parts of G. Therefore, the affine conjugate (and henceforth isomorphic) groups $G_\mu = \Phi_\mu \, G \, \Phi_\mu^{-1}$ *also act by rigid motions.* Since $L(G_\mu) = \Phi_\mu(L(G))$, Lemma 4.2 implies that G_μ violates 5.2 (i) if $\mu > 0$ is very small. Hence there exists a *minimal* $\mu' > 0$ such that $G_{\mu'}$ satisfies 5.2 (i). Since the affine transformations Φ_μ act trivially on E, the shortest vector in $L(G_{\mu'}) \backslash E$ must be a unit vector and $w_1, ..., w_k \in L(G_{\mu'})$. Now $G_{\mu'}$ has the required properties and Proposition 5.3 is proved.

5.4. *The proof* of Theorem II now proceeds in two steps.

Step 1. *Each normal crystallographic group G is uniquely characterized by a group table (*(ii) *below).*

Proof. Fix n linearly independent unit vectors $w_1, ... w_n \in L(G)$ and consider the sublattice

$$L = \{m_1 w_1 + ... + m_n w_n \mid m_1, ..., m_n \in \mathbf{Z}\}.$$

L is a subgroup of G. In each right coset modulo L of G we select a representative ω whose translation part w has length

(i) $$|w| \leqslant \frac{1}{2}(|w_1| + ... + |w_n|) = \frac{n}{2}.$$

Since G is discrete (2.4. (i)), there are only finitely many such representatives, say $\omega_{n+1}, ..., \omega_N$. Every $\alpha \in G$ can now be expressed in a unique way in the form

$$\alpha = (m_1 w_1 + ... + m_n w_n)\omega_\nu$$

where $n + 1 \leqslant \nu \leqslant N$. Since our L is isomorphic to $\mathbf{Z}^n$, G is uniquely determined (up to isomorphism) by the integers m_{ijk}, $\nu(j, k)$ and N which occur in the table

(ii) $$\omega_j \omega_k = (m_{1jk} w_1 + \ldots + m_{njk} w_n)\, \omega_{\nu(j,k)}, \quad j, k = 1, \ldots, N.$$

(For $i = 1, \ldots, n$, ω_i is the translation by w_i).

Clearly, the proof of Theorem II will be completed by

Step 2. The absolute values of m_{ijk}, $\nu(j, k)$ *and* N *in* (ii) *have an upper bound which depends only on the dimension* n *(see* (iii) *and* (iv) *below).*

Proof. The euclidean motions $\omega_{\nu(j,k)}$, ω_j and ω_k in (ii) have translation parts of length $\leqslant \frac{n}{2}$ (c.f. (i)). Consequently the translation $m_{1jk} w_1 + \ldots + m_{njk} w_n$ $= \omega_j \omega_k \omega_{\nu(j,k)}^{-1}$ has length $\leqslant \frac{3n}{2}$. In particular,

$$| m_{ijk} w_i^{\perp} | \leqslant \frac{3n}{2}, \quad i = 1, \ldots, n$$

where $w_i^{\perp}$ is the component of w_i perpendicular to the hyperplane E spanned by $w_1, \ldots, w_{i-1}, w_{i+1}, \ldots, w_n$. By Lemma 4.2 we have $| w_i^{\perp} | \geqslant (n+2)^{-n}$. Hence

(iii) $$| m_{ijk} | \leqslant \frac{3n}{2} (n+2)^n.$$

Now let us estimate N. The linear transformation $A = \operatorname{rot} \alpha$, $\alpha \in G$ is uniquely determined by its images $A(w_i)$, $i = 1, \ldots, n$. By Lemma 5.1 each of these images is a unit vector of $L(G)$ and, by Lemma 4.1, one out of at most 3^n candidates. It follows that at most $(3^n)^n$ different rotation parts occur in G.

If two elements ω_ρ and ω_σ among $\omega_{n+1}, \ldots, \omega_N$ have the same rotation part, then $\omega_\rho \omega_\sigma^{-1}$ is a vector of length $\leqslant \frac{n}{2} + \frac{n}{2}$ (c.f. (i)) and, again by Lemma 4.1, one out of at most $(2n+1)^n$ candidates. Hence

(iv) $$N \leqslant n + (3^n)^n \cdot (2n+1)^n.$$

Since $\nu(i, j) \leqslant N$, this concludes the proof of Theorem II.

5.5. *Remark.* From the preceding proof we can derive the upper bound $\exp \exp 4n^2$ for the number of isomorphism classes of n-dimensional crystallographic groups. The correct numbers for $n = 1, 2, 3, 4$ are respectively 2, 17, 219, 4783 [4].

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