3. Examples

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **31 (1985)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **26.05.2024**

This example is quite suggestive inasmuch as, in order to find a group relative to which the quotient Q is not co-complete, we have passed to a group which is large in comparison with Q. One might therefore speculate on the existence of quotient groups which are co-complete relative to all groups of a certain size. Examples of such quotients are presented in the next paragraph.

3. Examples

In view of (2.1), our examples are of superperfect groups Q whose homomorphic images of sufficiently small cardinality, say $\leqslant \alpha$, are all trivial. For this purpose it is worth recalling that an abelian group with a generating set of cardinality β has automorphism group of order at most 2^{β}. We feature three types of example.

I. *The acyclic groups considered by de la Harpe and McDuff*

Acyclic groups have the same homology (with trivial integer coefficients) as the trivial group and so are certainly superperfect. On the other hand, the acyclic groups discussed in [12] have the further property that any countable image is trivial. Hence they are *co-complete relative to all* K *with* $\mathrm{Out}(K)$ *countable*, and in particular relative to all finitely generated groups.

II. *The universal central extension over a simple group*

Let S be a non-abelian simple group. Being perfect, S admits a universal central extension Q [14], [17] (that is, an initial object in the category of all equivalence classes of extensions with central kernel and quotient S). Now Q is well-known to be superperfect — indeed, it is the unique superperfect central extension over S —, so we consider its possible images.

Proposition 3.1. *The non-trivial homomorphic images of* Q *are precisely the perfect central extensions of* S.

Since any image of Q is also perfect, clearly not all central extensions over S need be obtained in this way. For example, take the direct product of such an extension with an abelian group. However, if E has quotient S and central kernel then by [2 (1.6)b)] so does its maximal perfect subgroup $\mathcal{P}E$. So every central extension contains a preferred perfect central extension. In fact it contains a unique perfect central extension, because writing $\mathscr{P}E = P \, . \, C$ with P perfect and C central forces P to be normal in $\mathscr{P}E$ and the quotient $\mathscr{P}E/P$ abelian, hence trivial.

Of course the assertion, that any perfect central extension of (arbitrary) S is a homomorphic image of a superperfect one, generalises to the well-known result [11 p. 213] that any stem extension is an image of a stem cover. However this case admits an easy direct proof. For, given a central extension $D \overset{\iota}{\rightarrowtail} E \overset{\phi_E}{\twoheadrightarrow} S$ with E perfect, then the commuting triangle (which exists by universality)

$$\begin{tikzcd} & E \arrow[dd, "\phi_E"] \\ Q \arrow[ur, "\psi"] \arrow[dr, "\phi_Q"'] & \\ & S \end{tikzcd}$$

results in ψ being an epimorphism. For, any commutator $[e_1, e_2]$ in E is the image of $[q_1, q_2]$ in Q where $\phi_Q(q_i) = \phi_E(e_i)$, $i = 1, 2$. This is because $e_i \in \psi(q_i) \, . \, \iota(D)$ and

$$[\psi(q_1) \, . \, \iota(D), \psi(q_2) \, . \, \iota(D)] = [\psi(q_1), \psi(q_2)] \, .$$

It is the converse argument which uses the simplicity of S. Since all non-trivial quotients of Q are non-abelian, it suffices to check the following lemma.

LEMMA 3.2. *Let* E *be a central extension over* S. *Then*

(i) $S = E/Z(E)$, *and*

(ii) *every normal proper subgroup of* E *is central or contains* $[E, E]$. *Hence every non-abelian quotient of* E *is also a central extension over* S.

Proof. (i) Certainly $E/Z(E)$ is a quotient of S; it cannot be trivial since S is non-abelian.

(ii) Any normal subgroup N of E induces the normal subgroup $N \, . \, Z(E)/Z(E)$ of S. If N is non-central then this subgroup is non-trivial, hence S. Taking derived groups of the equation $E = N \, . \, Z(E)$ gives

$$[E, E] = [N, N] \leqslant N .$$

From (2.1) and (3.1) in combination we conclude immediately that the universal central extension Q over the non-abelian simple group S is *co-complete relative to all groups* K *such that no central extension over* S *is a subgroup of* $\mathrm{Out}(K)$.

For an important class of examples of this phenomenon, let F be any field. The Steinberg group $St_n(F)$ ($n \geqslant 3$, with $n = \infty$ representing $St(F)$, and with the groups $St_3(\mathbf{F}_2)$, $St_3(\mathbf{F}_4)$, $St_4(\mathbf{F}_2)$ excluded) is superperfect, being the universal central extension of the group $E_n(F)$ generated by elementary $n \times n$-matrices [17 p. 48]. Although this group is not simple (except for $E(F)$, by [1 V (2.1)]), its central quotient $PSL_n(F)$, the projective special linear group over F, is simple [1 V (4.1), (4.5)]. Hence (with the usual three exclusions), the Steinberg groups of a field are co-complete relative to all K whose $\mathrm{Out}(K)$ fails to contain any central extension of the corresponding projective special linear group. So, for example, $St_n(\mathbf{F}_q)$ must be co-complete relative to all K with

$$|\,\mathrm{Out}(K)\,| < |\,PSL_n(\mathbf{F}_q)\,| = (q^n-1)\,(q^n-q)\,...\,(q^n-q^{n-1})/(q-1)\,(n, q-1)\,.$$

III. *McLain groups*

First we recall the definition from [16], [19]. Let Λ be a linearly ordered set, F a field, and V a vector space over F with basis elements v_λ indexed by Λ. Then the McLain group $M(\Lambda, F)$ is the subgroup of the group of all linear transformations of V generated by elements of form $1 + ae_{\lambda\mu}$ where $a \in F$ and $\lambda, \mu \in \Lambda$ with $\lambda < \mu$. Here $e_{\lambda\mu}$ takes v_λ to v_μ and annihilates all other basis elements. For our purposes, it is more convenient to give an alternative description of $M(\Lambda, F)$ by means of a group presentation.

LEMMA 3.3. *The group* $M(\Lambda, F)$ *has presentation given by:*

generators

$$1 + ae_{\lambda\mu}\,, \qquad a \in F\,; \lambda, \mu \in \Lambda \quad \text{with} \quad \lambda < \mu$$

relations

$$(1+ae_{\lambda\mu})\,(1+be_{\lambda\mu}) = 1 + (a+b)e_{\lambda\mu} \qquad \text{(i)},$$

$$[1+ae_{\lambda\mu}, 1+be_{\zeta\eta}] = \begin{cases} 1 & \mu \neq \zeta, \lambda \neq \eta \qquad \text{(ii)}, \\ 1 + abe_{\lambda\eta} & \mu = \zeta \qquad \text{(iii)}. \end{cases}$$

Proof. The claimed relations follow quickly from the definitions, since $e_{\lambda\mu}e_{\zeta\eta} = e_{\lambda\eta}$ when $\mu = \zeta$ and is zero otherwise. To see that they imply all others, observe that any product which is not made trivial by these relations alone may be rewritten by means of (i), (ii), (iii) in the form

$$(1+ae_{\lambda_0\mu_0}) \prod_{\lambda>\lambda_0} (1+b_\lambda e_{\lambda\mu_0}) \prod_{\substack{\lambda \\ \mu<\mu_0}} (1+c_{\lambda\mu}e_{\lambda\mu})$$

for some λ_0, $\mu_0 \in \Lambda$ with $\lambda_0 < \mu_0$ and non-zero $a \in F$. However, the transformation represented by this product sends the basis element v_{λ_0} to a linear combination in which a appears as the coefficient of v_{μ_0}. Hence it is non-trivial. Thus $M(\Lambda, F)$ admits no relation which is not already a consequence of the given three types.

Despite obvious similarities with the Steinberg groups of II above, these groups are not accommodated by that discussion, for they are well-known to have trivial centre so long as Λ does not have a first and last element. Again, they are not perfect in general, unless Λ is dense. However, there is then the following further, somewhat surprising, similarity.

PROPOSITION 3.4. *If* Λ *is dense, then* $M(\Lambda, F)$ *is superperfect.*

The proof is deferred to the next section. An alternative (contemporaneous) proof, concentrating on the linear order structure of Λ, is to be found in [4].

PROPOSITION 3.5. *If* Λ *is dense, then the order of a non-trivial homomorphic image of* $M(\Lambda, F)$ *cannot be less than the cardinality of* F *or of every interval of* Λ.

Proof. Let π be an epimorphism from $M(\Lambda, F)$ onto a group of order less than card(F). Given an arbitrary generator $1 + ae_{\lambda\mu}$ of $M(\Lambda, F)$, take any ν in the interval (λ, μ) and consider the set $\{\pi(1+be_{\lambda\nu})\}_{b\in F}$. Since its cardinality is less than that of F, there exist distinct b_1, b_2 in F with $\pi(1+b_1e_{\lambda\nu}) = \pi(1+b_2e_{\lambda\nu})$. Then $1 + (b_1-b_2)e_{\lambda\nu}$ lies in Kerπ, whence so does

$$1 + ae_{\lambda\mu} = [1 + (b_1-b_2)e_{\lambda\nu}, 1 + (b_1-b_2)^{-1}ae_{\nu\mu}]\,.$$

The argument on the cardinality of intervals of Λ is similar (cf. [24 Lemma 1 (b)]).

The immediate conclusion of this discussion is that, for dense Λ, $M(\Lambda, F)$ *is co-complete relative to all groups* K *whose* Out(K) *has order less than the cardinality either of* F *or of every interval of* Λ.