Appendix — Homotopy groups of function spaces

Objekttyp: **Appendix**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **31 (1985)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **05.06.2024**

construct such a pair of factorisations. This construction combines with σ being well-defined, to yield (iii).

Hence σ is a homomorphism after all, and the proof is complete.

Appendix — Homotopy groups of function spaces

The aim here is to determine the homotopy type of certain function spaces which are needed in § 1 above. (All function spaces are to have the compact-open topology.) The literature on this topic is bedevilled by the requirement of local compactness of X for exponential correspondence between maps $W \times X \to Y$ and $W \to Y^X$. As a result much of it seems to divide into two camps: those who state the facts we need in unnecessary speciality, and those who, presumably having missed the point entirely, claim undue generality.

Fortunately, it is possible to vary the local compactness assumption a little (just enough): in the present context of studying homotopy groups, the space W will be a sphere, thus compact. Now exponential correspondence still holds whenever $W \times X$ is a k-space (compactly generated space), and for compact W this occurs whenever X itself is a k-space [8].

Proposition A.1. *Suppose that, for some* n, Y *has* $\pi_j(Y) = 0$ *for* $j > n$ *and* $\pi_1(Y)$ *acting nilpotently on* $\pi_j(Y)$ *for* $n - m < j \leqslant n$, *with* X *an* m*-connected* k*-space. Then, for all* $i \geqslant n - m$,

$$\pi_i\left((Y, y_0)^{(X, x_0)}\right) = 0 .$$

Proof. By the preceding remarks, we consider maps

$$S^i \times (X, x_0) \to (Y, y_0) .$$

Let $g: (X, x_0) \to (Y, y_0)$ lie in the relevant path-component of the function space. Then all maps $S^i \times X \to Y$ under investigation have to restrict to $* \vee g: S^i \vee X \to Y$. Obstructions to deforming an arbitrary extension $f: S^i \times X \to Y$ of $(* \vee g)$ to $g \circ pr_2: S^i \times X \to Y$ lie in the cohomology groups

$$H^q(S^i \times X, S^i \vee X; \{\pi_q(Y)\}) \cong \tilde{H}^{q-i}(X; \{\pi_q(Y)\}) .$$

Thus the only possible non-trivial obstructions lie in dimensions $q = i + 1, \ldots, n$. So when $m = 0$, $i \geqslant n$ suffices. More generally, let P_{n-m} be the first (lowest) space in the Postnikov tower for Y which is $(n-m)$-

equivalent to Y; then applying the special case shows that obstructions to the deformation $S^i \times X \to P_{n-m}$ vanish whenever $i \geqslant n - m$. Now $Y \to P_{n-m}$ is a nilpotent map [5] (by hypothesis). The obstructions to lifting this deformation to Y thus lie in groups of the form $\tilde{H}^{q-i}(X; G)$ where the coefficients are trivial and $n - m < q \leqslant n$. Finally, the assumption of m-connectivity of X ensures that these groups vanish for $q - i \leqslant m$, that is, $q \leqslant i + m = n$.

Since CW-complexes are compactly generated, we may certainly apply the above result; better, if X, Y merely have the homotopy type of CW-complexes then so does the function space, while the homotopy type of the function space is an invariant of that of X, Y. When combined with the Whitehead theorem, these remarks force the following.

COROLLARY A.2. *Let* X, Y *have the homotopy type of CW-complexes. Suppose that for some* $n \geqslant 1$, $\pi_j(Y) = 0$ *whenever* $j > n$ *and* $\pi_1(Y)$ *acts nilpotently on* $\pi_j(Y)$ *for* $2 \leqslant j \leqslant n$. *If* X *is* $(n-1)$*-connected, then each path-component of* $(Y, y_0)^{(X, x_0)}$ *is contractible.*

COROLLARY A.3. *For any group* K *and* $n \geqslant 1$ *(*K *abelian if* $n>1$*), let* $X = \mathscr{K}(K, n)$ *be an Eilenberg-Maclane space (*$\simeq$ *CW-complex). Then the path-component of* 1_X *in* $\mathscr{E}(X; x_0)$ *is contractible.*

In particular, for $n = 1$, this applies to the homotopy exact sequence of the homotopy fibration $\mathscr{E}(X; x_0) \to \mathscr{E}(X) \to X$ to show the triviality of the groups $\pi_i(\mathscr{E}(X))$, $i \geqslant 2$. The groups of path-components of $\mathscr{E}(X; x_0)$ and $\mathscr{E}(X)$ have been recorded in [6 p. 42], while $\pi_1(\mathscr{E}(X), 1_X)$ has been determined in [10] (albeit under conditions which our remarks have shown to be unnecessary). So in summary we have the following.

PROPOSITION A.4. *If* K *is a discrete group and* $X = \mathscr{K}(K, 1)$, *then* $\mathscr{E}(X; x_0)$ *has the homotopy type of the discrete group* $\mathrm{Aut}(K)$ *and* $\mathscr{E}(X)$ *has the homotopy type of* $\mathrm{Out}(K) \times \mathscr{K}(Z(K), 1)$.

The result needed for (1.1) above is obtained by taking the classifying space of the topological monoid $\mathscr{E}(X)$.

COROLLARY A.5. *With* K *and* X *as in* (A.4), *there is a fibration*

$$\mathscr{K}(Z(K), 2) \to B\mathscr{E}(X) \to \mathrm{BOut}(K).$$