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ON A CLASS OF ORTHOMODULAR QUADRATIC SPACES

by Herbert GROSS and Urs-Martin KÜNZI

0. INTRODUCTION

The most important property of the classical Hilbert space

$$\mathfrak{H} = \ell_2 = \{(\lambda_i)_{i\in\mathbf{N}} \mid \lambda_i \in \mathbf{R}\,,\ \Sigma\lambda_i^2 < \infty\}$$

is expressed by the projection theorem: the orthogonal complement $\mathfrak{X}^\perp$ of a closed linear subspace $\mathfrak{X}$ is a linear supplement, in formulae

$$(P_1) \qquad \mathfrak{X} = \overline{\mathfrak{X}} \Rightarrow \mathfrak{H} = \mathfrak{X} \oplus \mathfrak{X}^\perp$$

In the space $\mathfrak{H}$ it happens that precisely those linear subspaces $\mathfrak{X}$ are closed which coincide with their bi-orthogonals, $\mathfrak{X} = \overline{\mathfrak{X}} \Leftrightarrow \mathfrak{X} = (\mathfrak{X}^\perp)^\perp$ ("$\mathfrak{X}$ is $\perp$-closed"). Therefore we may express the projection theorem here in the following purely algebraic way

$$(P_2) \qquad \mathfrak{X} = (\mathfrak{X}^\perp)^\perp \Rightarrow \mathfrak{H} = \mathfrak{X} \oplus \mathfrak{X}^\perp$$

If, in the following, $\mathfrak{H}$ is any vector space over a division ring k and equipped with a hermitean form, then $\mathfrak{H}$ is called orthomodular if (P_2) holds for all linear subspaces $\mathfrak{X}$ of $\mathfrak{H}$.

The problem is to determine what orthomodular spaces there are besides classical Hilbert space (over $k = \mathbf{R}$, $\mathbf{C}$, $\mathbf{H}$). Notice that finite dimensional spaces are uninteresting in this connection because then validity of the projection theorem coincides with non-isotropy of the form. The first infinite dimensional orthomodular space different from the classical ones was discovered in 1979 by H. A. Keller [10 p. 3; 18].

We adduce the following motivations for the study of orthomodular spaces.

§ *1*. The requirement (P_2) on a hermitean space is an extraordinarily strong one. For years the endeavour of a number of people was directed towards proving that there are no examples other than classical Hilbert space [1, 9, 14, 25, 27, 33]. Indeed, all of the prominent Hilbert-like quadratic spaces discussed in the literature could be shown not to be orthomodular (See Sections II.1, II.2 below). As we now know a multitude of orthomodular spaces — there are examples for any characteristic of k — the question of what really lurks behind the projection theorem has become very interesting. The problem to determine *all* hermitean spaces with (P_2) is far from being solved. Although no topologies are involved in (P_2), all methods for the construction of orthomodular spaces that are known are based on topological considerations. The problem raises difficult questions concerning fields.

§ *2.* The Clifford algebras of certain orthomodular spaces H over k are ([6, 7]) normed k-algebras that are division rings ($*$-valued division rings in the sense of [14]). As the form on H has a canonical extension to its Clifford algebra (char $k \neq 2$), we obtain here a rather interesting class of division algebras that are infinite-dimensional over their centers. These division algebras, as hermitean spaces, are not orthomodular but they can be embedded into orthomodular spaces.

§ *3.* Let $\mathfrak{H}$ be Keller's space of [18] and $\mathscr{B}(\mathfrak{H})$ the algebra of bounded operators on $\mathfrak{H}$. There is hope to chance upon interesting rings of operators. Keller has given examples [19] of self-adjoint $A \in \mathscr{B}(\mathfrak{H})$ that share, among others, the following properties. The von Neumann algebra $\{A\}'$ (centralizer) is commutative; it is however — in contrast to the classical case — irreducible, A has no invariant subspaces. In these examples the arithmetic properties of k play a decisive role. One should first settle the problem whether all $\{A\}'$ with $A \in \mathscr{B}(\mathfrak{H})$ self-adjoint turn out commutative.

§ *4.* In the lattice theoretic viewpoint in physics introduced by G. Birkhoff and J. von Neumann ([4]) the experimentally verifiable propositions about a physical system are identified with the elements of an orthocomplemented lattice (Sec. I.2). On this lattice observables and states can be defined. In quantum physics one assumes that this lattice is the lattice $L_{\perp\perp}(\mathfrak{H})$ of an orthomodular space $\mathfrak{H}$ (or products of such lattices if super selection rules are present). If, for example, it could be made plausible that $\mathfrak{H}$ is over an archimedean ordered field and definite then by Theorem 5 $\mathfrak{H}$ would be a classical Hilbert space (as desired). In our opinion, the main use of Keller's discovery, as far as "quantum logic" is concerned, is to let the axiom that the logic be the usual Hilbert space structure appear even more *ad hoc* than is generally admitted. The base field of Keller's space $\mathfrak{H}$ is non-archimedean ordered. The frequently heard observation that scales on measuring devices in the laboratory are by necessity archimedean ordered is besides the point, for, scales are not connected with the division ring underlying the space $\mathfrak{H}$ but with the range $\mathbf{R}$ of the probability distributions

$$f : L_{\perp\perp}(\mathfrak{H}) \to [0, 1] \subset \mathbf{R}$$

that thrive on the lattice $L_{\perp\perp}(\mathfrak{H})$. Remarkably enough, there is a lavish supply of real valued probability distributions on $L_{\perp\perp}(\mathfrak{H})$ for our non-classical orthomodular spaces $\mathfrak{H}$ in spite of the teratological nature of the base fields (cf. Problem 7 in XIII). *Independent of any axiomatics there is the fascinating mathematical problem to classify these probability distributions.* No approach à la Gleason is possible here [8].

The present paper is meant as an introduction to the topic of orthomodular quadratic spaces. Attention is restricted to hermitean spaces $(\mathfrak{E}; \langle\ ,\ \rangle)$ over valued fields or ordered fields. Let $\mathscr{E}$ be the class of all spaces $\mathfrak{E}$ which admit a vector space topology that makes $\langle\ ,\ \rangle$ continuous (Section VIII). For expository purposes our main interest here is in the subclass $\mathscr{D} \subset \mathscr{E}$ of all "definite" spaces (Definition 15): these are the spaces $\mathfrak{E}$ where a norm defined on $\mathfrak{E}$ via the form $\langle\ ,\ \rangle$ and the valuation (ordering respectively) satisfies a Cauchy-Schwarz type inequality (Section IV). In both classes $\mathscr{D}$, $\mathscr{E}$ the spaces satisfying (P_1) can be characterized (Theorems 28, 34, 36); these spaces satisfy (P_2) as well. This characterization allows to construct orthomodular spaces at will.

We further give a survey of some older results related to orthomodular spaces (Section II). We also append a list of open problems.

I. Orthomodular spaces (Terminology)

I.1 Conventions for the whole paper: In this paper we consider left vector spaces $\mathfrak{E}$ over division rings k with involution $\alpha \mapsto \alpha^*$ (antiautomorphism of k whose square is the identity). $\mathfrak{E}$ is equipped with an anisotropic hermitean form $\langle\ ,\ \rangle$; thus by definition for all

$$\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in \mathfrak{E}, \alpha \in k:$$

$$\langle \alpha\mathfrak{a}+\mathfrak{b}, \mathfrak{c}\rangle = \alpha\langle \mathfrak{a}, \mathfrak{c}\rangle + \langle \mathfrak{b}, \mathfrak{c}\rangle, \langle \mathfrak{a}, \mathfrak{b}\rangle = \langle \mathfrak{b}, \mathfrak{a}\rangle^*, \langle \mathfrak{a}, \mathfrak{a}\rangle = 0 \quad \text{iff} \quad \mathfrak{a} = 0.$$

We shall often abreviate "$\langle \mathfrak{a}, \mathfrak{a}\rangle$" by "$\langle \mathfrak{a}\rangle$". If $\mathfrak{E}$ is infinite dimensional there are always subspaces $\mathfrak{F}$ that are properly contained in their biorthogonals $\mathfrak{F}^{\perp\perp} := (\mathfrak{F}^{\perp})^{\perp}$ [10; Lemma 3, p. 20]. Let $L(\mathfrak{E})$ be the set of all linear subspaces of $\mathfrak{E}$ and

(1) $$L_{\perp\perp}(\mathfrak{E}) := \{\mathfrak{F} \in L(\mathfrak{E}) \mid \mathfrak{F} = \mathfrak{F}^{\perp\perp}\}$$

We are interested in the set of splitting subspaces

(2) $$L_s(\mathfrak{E}) := \{\mathfrak{F} \in L(\mathfrak{E}) \mid \mathfrak{F} + \mathfrak{F}^{\perp} = \mathfrak{E}\}$$

Clearly $L_s(\mathfrak{E}) \subset L_{\perp\perp}(\mathfrak{E})$. A hermitean space $\mathfrak{E}$ is called *orthomodular* iff $L_s = L_{\perp\perp}$. In [6, 7, 9, 10, 18, 20; 31, 32] orthomodular spaces and forms were termed "hilbertian". However, "hilbertian form" already has a different meaning in the theory of normed algebras [5, Chap. XV.6] which actually causes equivocations. We have therefore yielded to the "orthomodular"-terminology.

In the following k is usually assumed to be a topological division ring and $\mathfrak{E}$ equipped with a vector space topology τ (which means that τ is compatible with the additive group of $\mathfrak{E}$ and scalar multiplication $k \times \mathfrak{E} \to \mathfrak{E}$ is continuous) such that the form $\langle\ ,\ \rangle$ on $\mathfrak{E}$ is (separately) continuous. We then consider the set of closed linear subspaces in $(\mathfrak{E}, \tau)$

(3) $$L_c(\mathfrak{E}) := \{\mathfrak{F} \in L(\mathfrak{E}) \mid \overline{\mathfrak{F}} = \mathfrak{F}\}$$

We have $L_{\perp\perp}(\mathfrak{E}) \subseteq L_c(\mathfrak{E})$ by continuity of the form.

Definition 1. The vector space topology τ on $\mathfrak{E}$ is admissible if and only if $L_{\perp\perp}(\mathfrak{E}) = L_c(\mathfrak{E})$.

Remark 2. All (infinite dimensional) orthomodular spaces $\mathfrak{E}$ discovered hitherto carry an admissible topology and this topology is needed to handle the space. Furthermore, all orthomodular spaces other than classical Hilbert space are separable in the sense that they contain *countable* families with $\perp$-dense span. This is quaint. No non-separable orthomodular space has been discovered so far. Cf. Remark 8.

I.2 Appendix on lattices. These brief remarks are not needed in order to understand the rest of the paper; however they throw light on concepts and related problems.

A *lattice* L is a non-void partially ordered set such that

$$a \vee b := \sup\{a, b\}, \quad a \wedge b := \inf\{a, b\}$$

exist for all pairs (and hence all finite sets) of elements of L. If arbitrary sets of elements of L admit suprema and infima then L is called *complete.* We always assume that L has *universal bounds* 0 and 1. An element b is said to *cover* an element a, $a < \cdot\, b$, when $a < b$ and for no c we have

$a < c < b$; *atoms* are elements that cover 0. A lattice is *atomistic* when every non-zero element a is the supremum of all atoms $\leqslant a$. The following property is *the covering property*: "if p is an atom and $a \wedge p = 0$ then $a <\cdot\, a \vee p$. Both $L(\mathfrak{E})$ and $L_{\perp\perp}(\mathfrak{E})$ are lattices with respect to $\subseteq$ whereas $L_s(\mathfrak{E})$ is not, in general, a lattice (cf. [9]). In fact, $L(\mathfrak{E})$ and $L_{\perp\perp}(\mathfrak{E})$ are complete, atomistic and they enjoy the covering property.

An *orthocomplementation* $a \mapsto a^\perp$ on a lattice L is a decreasing involution with $a^\perp \vee a = 1$, $a^\perp \wedge a = 0$. It follows that $(a \vee b)^\perp = a^\perp \wedge b^\perp$. An orthocomplemented lattice L is called *orthomodular* if its elements satisfy [15, p. 780]

$$a \leqslant b \Rightarrow b = a \vee (b \wedge a^\perp) \tag{4}$$

In an orthomodular lattice L we call *compatible* two elements a, b if $b = (b \wedge a) \vee (b \wedge a^\perp)$; this is the case iff the orthocomplemented lattice generated by a, b is distributive ([29, (2.25) p. 28]). If 0, 1 are the only elements compatible with all elements of L then L is called *irreducible*. A *propositional system* is a complete, orthomodular, atomistic lattice that enjoys the covering property.

The lattice $L_{\perp\perp}(\mathfrak{E})$ attached to a hermitean space is always orthocomplemented (recall that we assume the forms to be non-isotropic). If $\mathfrak{E}$ is orthomodular, then $L_{\perp\perp}(\mathfrak{E})$ is an orthomodular lattice, and conversely (hence the terminology). In fact, one easily verifies:

(5) If $L_s(\mathfrak{E}) = L_{\perp\perp}(\mathfrak{E})$ then $L_{\perp\perp}(\mathfrak{E})$ is an irreducible propositional system.

The following converse of (5) is essentially due to G. Birkhoff and J. v. Neumann [4, Appendix] and R. Baer [2, p. 302] (Cf. [10, p. 45], [23]).

THEOREM 3. *Let L be any irreducible propositional system of dimension $\geqslant 4$ (i.e. there is a chain $0 < a < b < c < d$ in L). Then L is $\perp$-isomorphic to the lattice $L_{\perp\perp}(\mathfrak{E})$ of some suitable orthomodular space $\mathfrak{E}$ over a suitable division ring k.*

This theorem explains the interest that the quantum logic approach to axiomatic quantum mechanics had taken in propositional systems: they lead towards the classical interpretation. The rub is that the division ring k need *not* be $\mathbf{R}$, $\mathbf{C}$ or $\mathbf{H}$ as we *know* since Keller's example [18]. In order to arrive at the classical structures stronger axioms on the lattice have to be postulated such as, for example, in [12, 33]. The reader interested in this kind of foundational problems in physics is refered to [3, 12, 15, 29].

Orthomodular lattices that derive from orthomodular quadratic spaces make up only a fraction of abstract orthomodular lattices (refer to [13, 16, 17]). The orthomodular law (4) is exceedingly enigmatic even if attention is restricted to orthomodular quadratic spaces. The complexity of the orthomodular conundrum does not surprise us anymore.

II. Results on orthomodular spaces prior to Keller's discovery

II.1. Results without topological restrictions on $\mathfrak{E}$. We begin with a classic ([1]).

Theorem 4 (Amemiya-Araki-Piron). *Let k be one of $\mathbf{R}, \mathbf{C}, \mathbf{H}$ and $\mathfrak{E}$ an infinite-dimensional k-vector space equipped with a positive definite hermitean form $\langle\ ,\ \rangle$ (relative to the usual involution $*$ in k). Then $\mathfrak{E}$ is orthomodular iff $\mathfrak{E}$ is complete as a normed space*

$$(\|\mathfrak{x}\| := \langle \mathfrak{x}, \mathfrak{x}\rangle^{\frac{1}{2}}),$$

i.e. iff $\mathfrak{E}$ is a Hilbert space.

If, in the setting of Thm. 4, we pass to subfields of k then the same conclusion can be drawn although the proof is much more tricky [9]:

Theorem 5 (Gross-Keller). *Let k be an archimedean (Baer-)ordered $*$-field* ([14, p. 219]) *and $\mathfrak{E}$ an infinite dimensional k-vector space equipped with a positive definite hermitean form. Then the following are equivalent*

(i) *k is one of $\mathbf{R}, \mathbf{C}, \mathbf{H}$ and $\mathfrak{E}$ is a Hilbert space*

(ii) $L_s(\mathfrak{E}) = L_{\perp\perp}(\mathfrak{E})$ *i.e. $\mathfrak{E}$ is orthomodular*

(iii) $L_c(\mathfrak{E}) = L_{\perp\perp}(\mathfrak{E})$ *(c refers to the norm $\| \mathfrak{x} \| := \langle \mathfrak{x}, \mathfrak{x}\rangle^{\frac{1}{2}} \in k^{\frac{1}{2}}$)*

(iv) $L_s(\mathfrak{E}) = L_{\perp\perp}(\mathfrak{E}) = L_c(\mathfrak{E})$.

Remark 6. In [24] sequence spaces $\mathfrak{E} := \ell_2(k)$ for $k \subset \mathbf{H}$ are considered and equipped with hermitean maps (not forms) $\mathfrak{E} \times \mathfrak{E} \to \mathbf{H}$. Again, the lattice of $\perp$-closed subspaces in $\mathfrak{E}$ is orthomodular iff $k = \mathbf{R}, \mathbf{C}$, or $\mathbf{H}$.

Another attempt to chance upon new orthomodular forms is to replace the reals by the non-archimedian ordered field $^*\mathbf{R}$, a non-standard model of $\mathbf{R}$. However [28]:

THEOREM 7 (Morash). *The inner product on* $\mathfrak{H} = \ell_2(\mathbf{R})$ *induces a positive definite symmetric bilinear form* $*\mathfrak{H} \times *\mathfrak{H} \to *\mathbf{R}$; *here* $*\mathfrak{H}$ *is the set (linear* $*\mathbf{R}$*-space) of equivalence classes in* $\mathfrak{H}^{\mathbf{N}}$ *induced by the free ultra filter* U *on* $\mathbf{N}$ *used to define* $*\mathbf{R}$. *The lattice* $L_{\perp\perp}(*\mathfrak{H})$ *is complete but not orthomodular.*

Remark 8. In [28] it is also shown that the ultra filter construction applied to a product of lattices isomorphic to $L_{\perp\perp}(\ell_2(\mathbf{R}))$ leads to an orthomodular lattice that, alas, is not complete. This loss of completeness, incidentally, is *the* (only) obstacle on the way to an easy (ultrafilter construction + Theorem 3) existence proof for orthomodular spaces different from Hilbert space.

A rather general theorem is ([33]):

THEOREM 9 (Wilbur). *Let* $(k, *)$ *be commutative and such that for each *-symmetric element* $\lambda \in k$ *there is* $\alpha \in k$ *with* $\lambda = \pm\alpha\alpha^*$. *If* $\mathfrak{E}$ *is an orthomodular space over* k, $\dim \mathfrak{E}$ *infinite, then* $k = \mathbf{R}$ *or* $\mathbf{C}$ *with* $*$ *the identity or the usual conjugation, respectively (so* $\mathfrak{E}$ *is a Hilbert space).*

Remark 10. The formulation of Thm. 9 in [33] also admits skew $(k, *)$ with one additional assumption. However, by Dieudonné's Lemma ([10 p. 18]) $(k, *)$ must then be a quaternion algebra with $*$ the usual conjugation.

Wilbur's result is generalized to ordered *-fields in [14, § 6].

Hermitean spaces that are orthogonal sums of finite dimensional subspaces are called *diagonal*; subspaces of diagonal spaces are termed *prediagonal.* There is a full-fledged theory about prediagonal spaces of infinite dimensions. Deplorably, we have ([9]):

THEOREM 11 (Gross-Keller). *Let* $\dim \mathfrak{E} \geqslant \aleph_0$. *If* $\mathfrak{E}$ *is prediagonal then it is not orthomodular. Thus, in particular,* $\dim \mathfrak{E} > \aleph_0$ *if* $\mathfrak{E}$ *is orthomodular.*

Orthomodularity of a space $\mathfrak{E}$ has strange consequences for the base field of $\mathfrak{E}$. We just mention one of several [9, p. 15].

THEOREM 12 (Gross-Keller). *If* $\operatorname{card} k < 2^{\aleph_0}$ *then an infinite dimension al k-space* $\mathfrak{E}$ *cannot be orthomodular.*

II.2. A RESULT ON SPACES $\mathfrak{E}$ EQUIPPED WITH AN ADMISSIBLE TOPOLOGY. Certain well known classes of spaces $\mathfrak{E}$ that carry admissible topologies can be proved *not* to contain orthomodular specimen; we refer to [9]. Here we mention but one result ([9, p. 20]); it has been crucial on the road to Keller's discovery. The idea of its proof is used again in the proof of Theorem 17 below.

THEOREM 13 (Gross-Keller). *Let* k *be a non archimedean ordered field and equipped with its order topology; let* $\langle\ ,\ \rangle$ *be a definite symmetric form on the k-vector space* $\mathfrak{E}$. *Equip* $\mathfrak{E}$ *with the norm topology*

$$(\|\mathfrak{x}\| := \langle \mathfrak{x}, \mathfrak{x}\rangle^{\frac{1}{2}} \in k^{\frac{1}{2}}).$$

Assume that $\mathfrak{E}$ *contains at least one orthogonal family* $(\mathfrak{e}_i)_{i\in\mathbf{N}}$ *that is bounded, i.e. for suitable* $\alpha, \beta \in k$

$$0 < \alpha \leqslant \langle \mathfrak{e}_i, \mathfrak{e}_i \rangle \leqslant \beta \quad (i\in\mathbf{N}) \tag{6}$$

Then $L_{\perp\perp}(\mathfrak{E}) \underset{\neq}{\subset} L_c(\mathfrak{E})$.

III. KELLER'S EXAMPLE

The authors of [9] lamented about the "irksome" condition (6) which, indeed, need not be satisfied (*loc. cit.*, p. 89). Keller finally noticed that (6) pointed at the very crux of the matter. He considered the transcendental extension $k_0 = \mathbf{Q}(X_i)_{i\in\mathbf{N}}$ with the unique ordering that has $X_0 > q$ for all $q \in \mathbf{Q}$ and $X_i^n < X_{i+1}$ for all i and all n; then he let k be the completion of k_0 by means of Cauchy sequences. $\mathfrak{E}$ is the linear k-space of all $(y_i)_{i\in\mathbf{N}} \in k^{\mathbf{N}}$ such that $\sum_{\mathbf{N}} y_i^2 X_i$ exists (addition and scalar multiplication component wise) and $\langle (y_i)_{i\in\mathbf{N}}, (z_i)_{i\in\mathbf{N}} \rangle := \sum_{\mathbf{N}} y_i z_i X_i$. Original and ingenious arguments given in [18] establish orthomodularity of $\mathfrak{E}$. (This also follows from our Theorem 36 below.)

Gross noticed that Keller's construction works for valued fields ([6, 7, 20]). An example is also contained in [14, p. 237]).

Keller's choice of a field over which one can build orthomodular spaces has been good: as our results show his space exhibits the typical properties of an orthomodular space with an admissible topology (cf. Remark 29 below).

IV. THE FUNDAMENTAL INEQUALITIES IN DEFINITE SPACES

IV.1. *-VALUATIONS (cf. [14]). Let $(k, *)$ be an involutorial division ring and Γ a totally ordered (additively written) abelian group. A surjective map

$$\varphi : k \to \Gamma \cup \{\infty\} \qquad (a + \infty = \infty \quad \text{for all} \quad a \in \Gamma \cup \{\infty\}) \tag{7}$$

is called *-valuation iff (i) $\varphi(x+y) \geqslant \min\{\varphi(x), \varphi(y)\}$, (ii) $\varphi(xy) = \varphi(x) + \varphi(y)$, (iii) $\varphi(x) = \infty \Leftrightarrow x = 0$, (iv) $\varphi(x) = \varphi(x^*)$.

The set of all $U_\varepsilon := \{x \in k \mid \varphi(x) \geqslant \varepsilon\}$, $\varepsilon \in \Gamma$, is a neighbourhood basis for a division ring topology on k. In general we think of $(k, *)$ as equipped with this topology.

IV.2. THE INEQUALITIES. Assume that char $k \neq 2$ and that the valuation in (7) has $\varphi(2) = 0$ (cf. Remark 35). Let $\langle\ ,\ \rangle$ be a hermitean form on a k-space $\mathfrak{E}$. Assume $\mathfrak{E}$ non-degenerate $(\mathfrak{E}^\perp = (0))$. Recall that we write "$\langle \mathfrak{x} \rangle$" for $\langle \mathfrak{x}, \mathfrak{x} \rangle$, $\mathfrak{x} \in \mathfrak{E}$. It is useful to know a proof for the following fact

LEMMA 14 ([20]). *The following four statements are equivalent*

(i) $\forall \mathfrak{x}, \mathfrak{y} \in \mathfrak{E} : \varphi\langle \mathfrak{x}+\mathfrak{y} \rangle \geqslant \min\{\varphi\langle \mathfrak{x} \rangle, \varphi\langle \mathfrak{y} \rangle\}$ *(triangle inequality)*

(ii) $\forall \mathfrak{x}, \mathfrak{y} \in \mathfrak{E} : \langle \mathfrak{x}, \mathfrak{y} \rangle = 0 \Rightarrow \varphi\langle \mathfrak{x}+\mathfrak{y} \rangle = \min\{\varphi\langle \mathfrak{x} \rangle, \varphi\langle \mathfrak{y} \rangle\}$ *("Pythagoras")*

(iii) $\forall \mathfrak{x}, \mathfrak{y} \in \mathfrak{E} : \varphi\langle \mathfrak{x}, \mathfrak{y} \rangle \geqslant \min\{\varphi\langle \mathfrak{x} \rangle, \varphi\langle \mathfrak{y} \rangle\}$ *("weak Cauchy-Schwarz")*

(iv) $\forall \mathfrak{x}, \mathfrak{y} \in \mathfrak{E} : 2\varphi\langle \mathfrak{x}, \mathfrak{y} \rangle \geqslant \varphi\langle \mathfrak{x} \rangle + \varphi\langle \mathfrak{y} \rangle$ *("Cauchy-Schwarz")*

(Notice that each statement implies anisotropy of $\mathfrak{E}$*).*

Proof. (i) $\Rightarrow$ (ii): Let $\mathfrak{x} \perp \mathfrak{y}$ and

$$\varphi\langle \mathfrak{x} \rangle \leqslant \varphi\langle \mathfrak{y} \rangle; \quad \varphi\langle \mathfrak{x} \rangle = \varphi\langle 2\mathfrak{x} \rangle = \varphi\langle (\mathfrak{x}+\mathfrak{y})$$
$$+ (\mathfrak{x}-\mathfrak{y}) \rangle \geqslant \min\{\varphi\langle \mathfrak{x}+\mathfrak{y} \rangle, \varphi\langle \mathfrak{x}-\mathfrak{y} \rangle\} = \varphi\langle \mathfrak{x}+\mathfrak{y} \rangle \geqslant \varphi\langle \mathfrak{x} \rangle .$$

(ii) $\Rightarrow$ (iv): Assume $\mathfrak{x} \neq 0 \neq \mathfrak{y}$. Pick $\mathfrak{b}$ in the span of $\mathfrak{x}, \mathfrak{y}$ such that

$$\mathfrak{x} = \mathfrak{b} + \lambda\mathfrak{y}, \ \mathfrak{b} \perp \mathfrak{y}; \quad 2\varphi\langle \mathfrak{x}, \mathfrak{y} \rangle = 2\varphi\langle \mathfrak{b}+\lambda\mathfrak{y}, \mathfrak{y} \rangle = 2\varphi\langle \lambda\mathfrak{y}, \mathfrak{y} \rangle$$
$$= 2\varphi(\lambda) + 2\varphi\langle \mathfrak{y} \rangle = \varphi\langle \lambda\mathfrak{y} \rangle + \varphi\langle \mathfrak{y} \rangle \geqslant \varphi\langle \mathfrak{x} \rangle + \varphi\langle \mathfrak{y} \rangle .$$

(iv) $\Rightarrow$ (iii): trivial

(iii) $\Rightarrow$ (i): straight forward. $\square$

IV.3. THE CLASS $\mathscr{D}$ OF DEFINITE SPACES. Positive definite forms over ordered fields satisfy the triangle inequality as well as the Cauchy-Schwarz inequality. We therefore set down

Definition 15. A definite space is a nondegenerate hermitean space $(\mathfrak{E}; \langle\ ,\ \rangle)$ over an involutorial division ring $(k, *)$, char $k \neq 2$, that is equipped with a $*$-valuation φ that has $\varphi(2) = 0$ (cf. Remark 35) and that satisfies one (and hence all) of the four statements in Lemma 14. A definite space $\mathfrak{E}$ will always be considered as a topological vector space, the topology being given by the zero-neighbourhood basis $\mathfrak{U}_\gamma := \{\mathfrak{y} \in \mathfrak{E} \mid \varphi\langle\mathfrak{y}\rangle \geqslant \gamma\}, \gamma \in \Gamma$. If $(\mathfrak{e}_\iota)_{\iota\in I}$ is any family over vectors in $\mathfrak{E}$ such that the net of all *finite* ("partial") sums $\sum \mathfrak{e}_\iota$ has a limit $\mathfrak{x}$ in $\mathfrak{E}$ then we write $\mathfrak{x} = \sum_{\iota\in I} \mathfrak{e}_\iota$ and call $(\mathfrak{e}_\iota)_{\iota\in I}$ summable.

LEMMA 16. *Let* $(\mathfrak{e}_\iota)_{\iota\in I}$ *be an orthogonal family in the definite space* $(\mathfrak{E}; \langle\ ,\ \rangle)$ *and* $\mathfrak{F}$ *its span. For each* $\mathfrak{x}$ *in the topological closure of* $\mathfrak{F}$ *we have* $\mathfrak{x} = \sum_{\iota\in I} \langle\mathfrak{x}, \mathfrak{e}_\iota\rangle \langle\mathfrak{e}_\iota\rangle^{-1}\mathfrak{e}_\iota$.

Proof. Let $\mathscr{P}$ be the set of all finite subsets of I. For $V \in \mathscr{P}$ we set $\mathfrak{x}_V := \sum_{\iota\in V} \langle\mathfrak{x}, \mathfrak{e}_\iota\rangle \langle\mathfrak{e}_\iota\rangle^{-1}\mathfrak{e}_\iota$. We have to prove that for each $\gamma \in \Gamma$ there is $U \in \mathscr{P}$ such that $\varphi\langle\mathfrak{x}-\mathfrak{x}_V\rangle \geqslant \varepsilon$ for all V *with* $U \subset V \in \mathscr{P}$. Now there is $\mathfrak{y} \in \mathfrak{F}$ with $\varphi(\mathfrak{x}-\mathfrak{y}\rangle \geqslant \varepsilon$. Pick $U \in \mathscr{P}$ with $\mathfrak{y} \in \operatorname{span}\{\mathfrak{e}_\iota \mid \iota \in U\}$. If $U \subset V \in \mathscr{P}$ then $\mathfrak{x} - \mathfrak{x}_V \perp \mathfrak{x}_V - \mathfrak{y}$, so by "Pythagoras" (Lemma 14 (ii)) we obtain $\varepsilon \leqslant \varphi\langle\mathfrak{x}-\mathfrak{y}\rangle = \min\{\varphi\langle\mathfrak{x}-\mathfrak{x}_V\rangle, \varphi\langle\mathfrak{x}_V-\mathfrak{y}\rangle\} \leqslant \varphi\langle\mathfrak{x}-\mathfrak{x}_V\rangle$. □

V. NECESSARY CONDITIONS IN $\mathscr{D}$ FOR $L_c = L_{\perp\perp}$

The principal result of this section is

THEOREM 17 ([20]). *Let* $\mathfrak{E}$ *be an infinite dimensional definite space carrying an admissible topology i.e., the topology mentioned in Definition 15 is admissible in the sense of Definition 1; let furthermore* $(\mathfrak{e}_\iota)_{\iota\in I}$ *be an orthogonal family in* $\mathfrak{E}$ *such that* $(\varphi\langle\mathfrak{e}_\iota\rangle)_{\iota\in I}$ *has a lower bound in* Γ. *Then* $\sum_{\iota\in I} \mathfrak{e}_\iota$ *exists.*

Proof. Let $\mathfrak{F} := \operatorname{span}\{\langle\mathfrak{e}_\iota\rangle^{-1}\mathfrak{e}_\iota - \langle\mathfrak{e}_0\rangle^{-1}\mathfrak{e}_0 \mid \iota \in I\}$. We first wish to show that $\langle\mathfrak{e}_0\rangle^{-1}\mathfrak{e}_0$ is not an element of the topological closure $\overline{\mathfrak{F}}$. Indeed, if γ is a lower bound of $(\varphi\langle \mathfrak{e}_\iota\rangle)_{\iota\in I}$ and if we let $\mathfrak{x} := \sum_{\iota\in U} \lambda_\iota(\langle \mathfrak{e}_\iota\rangle^{-1}\mathfrak{e}_\iota - \langle \mathfrak{e}_0\rangle^{-1}\mathfrak{e}_0)$ be a typical vector of $\mathfrak{F}$ (U some finite nonvoid subset of $I\backslash\{0\}$) then we get the inequalities

$$\begin{aligned}\varphi\langle \mathfrak{x} - \langle \mathfrak{e}_0\rangle^{-1}\mathfrak{e}_0\rangle &= \varphi\langle(-1-\sum_U \lambda_\iota)\langle \mathfrak{e}_0\rangle^{-1}\mathfrak{e}_0 + \sum \lambda_\iota <\mathfrak{e}_\iota>^{-1}\mathfrak{e}_\iota\rangle \\ &= \min_{\iota\in U} \{2\varphi(-1-\sum_U \lambda_\iota) - \varphi\langle \mathfrak{e}_0\rangle, 2\varphi(\lambda_\iota) - \varphi\langle \mathfrak{e}_\iota >)\} \\ &\leqslant 2\min_{\iota\in U} \{\varphi(-1-\sum_U \lambda_\iota), \varphi(\lambda_\iota)\} - \gamma \leqslant \varphi(-1) - \gamma = -\gamma.\end{aligned}$$

Thus $\overline{\mathfrak{F}} \neq \mathfrak{E}$.

Since $\mathfrak{F}^{\perp\perp} = \overline{\mathfrak{F}}$ we have $\mathfrak{F}^\perp \neq (0)$. Pick a non-zero $\mathfrak{x} \in \mathfrak{F}^\perp$; so $\langle \mathfrak{x}, \mathfrak{e}_\iota\rangle \langle \mathfrak{e}_\iota\rangle^{-1} = \langle \mathfrak{x}, \mathfrak{e}_0\rangle \langle \mathfrak{e}_0\rangle^{-1}$. If we assume that $(\mathfrak{e}_\iota)_{\iota\in I}$ is a *maximal* orthogonal family then by $L_c = L_s$ and Lemma 16 $\mathfrak{x} = \sum_I \langle \mathfrak{x}, \mathfrak{e}_\iota\rangle \langle \mathfrak{e}_\iota\rangle^{-1}\mathfrak{e}_\iota = \langle \mathfrak{x}, \mathfrak{e}_0\rangle \langle \mathfrak{e}_0\rangle^{-1} \sum_I \mathfrak{e}_\iota$ and thus $\sum_I \mathfrak{e}_\iota \in \mathfrak{F}^\perp$. If $(\mathfrak{e}_\iota)_{\iota\in I}$ is not maximal then we write it as a difference of two maximal bounded families: Complete the given family to a maximal orthogonal bounded family $(\mathfrak{e}_\iota)_{\iota\in J}$, $J \supset I$, by Zorn's Lemma. For $\iota \in J$ let $\alpha_\iota := 1 \in k$ when $\iota \in I$ and $\alpha_\iota := 2$ when $\iota \in J\backslash I$. The two families $(2\mathfrak{e}_\iota)_{\iota\in J}$, $(\alpha_\iota\mathfrak{e}_\iota)_{\iota\in J}$ are bounded maximal families to which the previous result may be applied. We get $\sum_{\iota\in I} \mathfrak{e}_\iota = \sum_{\iota\in I} (2\mathfrak{e}_\iota) - \sum_{\iota\in I} \alpha_\iota\mathfrak{e}_\iota \in \mathfrak{E}$. □

COROLLARY 18. *If* $\mathfrak{E}$ *and* $(\mathfrak{e}_\iota)_{\iota\in I}$ *are as in Theorem 17 then* $(\mathfrak{e}_\iota)_{\iota\in I}$ *converges to* $0 \in \mathfrak{E}$. □

COROLLARY 19. *If* $\mathfrak{E}$ *is as in Theorem 17 then the cofinality type of* Γ *is* ω_0. *In particular, the topology on* $\mathfrak{E}$ *satisfies the first countability axiom.* □

COROLLARY 20. *If* $\mathfrak{E}$ *is as in Theorem 17 then all orthogonal families of non-zero vectors are countable.*

Proof. Let $(\mathfrak{e}_\iota)_{\iota\in I}$ be such a family; by multiplying $\mathfrak{e}_\iota$ by a suitable scalar, if necessary, we may assume $(\varphi\langle \mathfrak{e}_\iota\rangle)_{\iota\in I}$ to be bounded below. Since $\sum_{\iota\in I} \mathfrak{e}_\iota$ exists by Theorem 17, the sets $I_\gamma = \{\iota \in I \mid \varphi\langle \mathfrak{e}_\iota\rangle \leqslant \gamma\}$ are finite for all $\gamma \in \Gamma$. Let $(\gamma_i)_{i\in\mathbf{N}}$ be confinal in Γ. Then $I = \cup\{I_{\gamma_i} \mid i \in \mathbf{N}\}$ is countable. □

Definition 21. The elements of the group $\Gamma/2\Gamma$ are called *types.* Let $T: \Gamma \to \Gamma/2\Gamma$ be the canonical projection. $T \circ \varphi$ is constant on the square classes of k (elements of $\dot{k}/\dot{k}^2$) and $T \circ \varphi \circ \langle\ \rangle$ is constant on the "punctured" straight lines in E. A family $(\mathfrak{e}_\iota)_{\iota\in I}$ of vectors in $\mathfrak{E}$ is said to satisfy the *type-condition* iff for all $(\alpha_\iota)_{\iota\in I} \in k^I$ the following holds: if $(\varphi\langle\alpha_\iota \mathfrak{e}_\iota\rangle)_{\iota\in I}$ is bounded (below) then $(\alpha_\iota \mathfrak{e}_\iota)_{\iota\in I}$ converges to $0 \in E$.

COROLLARY 22. *Let $\mathfrak{E}$ be as in Theorem 17. $\Gamma/2\Gamma$ is infinite. Each orthogonal family in $\mathfrak{E}$ satisfies the type-condition, equivalently, $\Gamma/2\Gamma$ satisfies (8) below.* □

COROLLARY 23. *Let $\mathfrak{E}$ be as in Theorem 17. Then k is complete.*

Proof. By Corollary 19 it suffices to show that a sequence $(\alpha_i)_{i\in\mathbf{N}}$ with limit $0 \in k$ is summable. Let $(\mathfrak{e}_i)_{i\in\mathbf{N}}$ be maximal orthogonal in $\mathfrak{E}$ with $(\varphi\langle\mathfrak{e}_i\rangle)_{i\in\mathbf{N}}$ bounded below. If $(\lambda_i)_{i\in\mathbf{N}} \in k^{\mathbf{N}}$ has $(\varphi(\lambda_i))_{i\in\mathbf{N}}$ bounded below then $(\lambda_i \mathfrak{e}_i)_{i\in\mathbf{N}}$ is summable and by continuity of $\langle\ ,\ \rangle$ we obtain

$$\left\langle \sum_{\mathbf{N}} \lambda_i \mathfrak{e}_i, \sum_{\mathbf{N}} \mathfrak{e}_i \right\rangle = \sum_{\mathbf{N}} \lambda_i \langle \mathfrak{e}_i \rangle .$$

Thus, all families $(\lambda_i\langle\mathfrak{e}_i\rangle)_{i\in\mathbf{N}}$ with bounded $(\lambda_i)_{i\in\mathbf{N}}$ are summable.

Pick a strictly monotonic sequence $(n_i)_{i\in\mathbf{N}} \in \mathbf{N}^{\mathbf{N}}$ with $u_0 = 0$ and for all $i \in \mathbf{N}^+$ and all $m \geqslant n_i$: $\varphi(\alpha_m) > \varphi\langle\mathfrak{e}_i\rangle$, and set $A_i := \sum\{\alpha_j \mid n_i \leqslant j < n_{i+1}\}$. The family $(A_i)_{i\in\mathbf{N}}$ is summable if and only if $(\alpha_i)_{i\in\mathbf{N}}$ summable and, if the sums exist, these must be equal. If we set $\lambda_i := A_i\langle\mathfrak{e}_i\rangle^{-1}$ then, by what we have shown, the family of the $A_i = \lambda_i\langle\mathfrak{e}_i\rangle$ is summable. □

COROLLARY 24. *Let $\mathfrak{E}$ be as in Theorem 17. Then $\mathfrak{E}$ is complete.*

Proof. Let $(\mathfrak{x}_i)_{i\in\mathbf{N}}$ be a Cauchy sequence (Corollary 19). For each fixed $\mathfrak{y} \in \mathfrak{E}$ the map $\mathfrak{x} \mapsto \langle\mathfrak{y}, \mathfrak{x}\rangle$ is uniformly continuous. Hence by Cor. 23 the map $f : \mathfrak{y} \mapsto \lim_i \langle\mathfrak{y}, \mathfrak{x}_i\rangle$ is well-defined. As it is a continuous linear map, its kernel is a closed hyper-plane and so $(L_c(\mathfrak{E}) = L_{\perp\perp}(\mathfrak{E}))$ there is $\mathfrak{a} \in \mathfrak{E}$ such that $f(\mathfrak{y}) = \langle\mathfrak{x}, \mathfrak{a}\rangle$. Let $N \subseteq \mathbf{N}$ be infinite. Because $\lim \varphi\langle\mathfrak{y}, \mathfrak{a} - \mathfrak{x}_i\rangle = \infty$ for all $\mathfrak{y} \in \mathfrak{E}$ it follows by systematic use of the Cauchy-Schwarz inequality that $\{\varphi\langle\mathfrak{a} - \mathfrak{x}_i\rangle \mid i \in N\}$ is not bounded above by any $\gamma \in \Gamma$. Therefore $(\mathfrak{x}_i)_{i\in\mathbf{N}}$ converges to $\mathfrak{a}$.

VI. SUFFICIENT CONDITIONS IN $\mathscr{D}$ FOR $L_c = L_{\perp\perp}$

VI.1. ASSUMPTIONS. In this chapter $(\mathfrak{E}; \langle\ ,\ \rangle)$ is a definite space in the sense of Definition 15. Of the base field k we shall furthermore assume

(cf. Corolllaries 22 and 23)

$\Gamma/2\Gamma$ contains a sequence $(\xi_i+2\Gamma)_{i\in\mathbf{N}}$ such that each

(8) system of representatives $(\xi_i+2\gamma_i)_{i\in\mathbf{N}}$ that is bounded below tends to ∞.

(9) k is complete.

Thus, by (8), $\Gamma/2\Gamma$ will be infinite and the topology on k will satisfy the first countability axiom. There are many fields that satisfy (8) and (9): See Remark 30.

The results in the next sections will culminate in Theorem 28 which characterizes certain definite spaces that are orthomodular.

VI.2. COUNTING TYPES. Let $\mathfrak{E}$ be the completion of an $\aleph_0$-dimensional space $\mathfrak{F}$ which is spanned by an orthogonal basis $\mathscr{B} = (\mathfrak{e}_i)_{i\in\mathbf{N}}$ that satisfies the type condition (Def. 21). $\mathfrak{F}$ is dense in $\mathfrak{E}$ so $\mathfrak{F}^{\perp} = (0)$ and hence $\mathscr{B}$ is maximal. By Lemma 16 we have therefore $\mathfrak{x} = \sum_{\mathbf{N}} \langle \mathfrak{x}, \mathfrak{e}_i\rangle \langle \mathfrak{e}_i\rangle^{-1}\mathfrak{e}_i$ for all $\mathfrak{x}\in\mathfrak{E}$.

We now introduce the function ν which counts types on $\mathscr{B}$. Let $\nu\colon \Gamma/2\Gamma \to \mathbf{N}\colon t \mapsto \operatorname{card}\{i\in\mathbf{N} \mid T\circ\varphi\langle\mathfrak{e}_i\rangle = t\}$ (cf. Def. 21). We have

LEMMA 25. *If* $\mathfrak{f}_1, \ldots, \mathfrak{f}_m$ *are pairwise orthogonal (non zero) vectors in* $\mathfrak{E}$ *with* $T\circ\varphi\langle\mathfrak{f}_i\rangle = t\in\Gamma/2\Gamma$ *for all* $1\leqslant i\leqslant m$ *then* $m\leqslant \nu(t)$.

Proof. We shall replace the $\mathfrak{f}_i$ by suitable multiples and assume that $\varphi\langle\mathfrak{f}_i\rangle = \gamma\in\Gamma$ for all $1\leqslant i\leqslant m$. Let $J := \{i\in\mathbf{N} \mid T\circ\varphi\langle\mathfrak{e}_i\rangle = t\}$. We have $\mathfrak{f}_j = \mathfrak{f}_j' + \mathfrak{f}_j''$ where

$$\mathfrak{f}_j' := \sum_J \langle\mathfrak{f}_j, \mathfrak{e}_i\rangle\langle\mathfrak{e}_i\rangle^{-1}\mathfrak{e}_i, \qquad \mathfrak{f}_j'' := \sum_{\mathbf{N}\setminus J}\langle\mathfrak{f}_i, \mathfrak{e}_i\rangle\langle\mathfrak{e}_i\rangle^{-1}\mathfrak{e}_i\,.$$

Since Lemma 14 (ii) generalizes to finite as well as to infinite sums we find $\varphi\langle\mathfrak{f}_j''\rangle = \min_{i\in\mathbf{N}\setminus J}\{\varphi\langle\langle\mathfrak{f}_j,\mathfrak{e}_i\rangle\langle\mathfrak{e}_i\rangle^{-1}\mathfrak{e}_i\rangle\} \neq \varphi\langle\mathfrak{f}_j\rangle$ (because types are different). By Lemma 14 (ii) furthermore $\varphi\langle\mathfrak{f}_j\rangle \leqslant \varphi\langle\mathfrak{f}_j'\rangle$, $\varphi\langle\mathfrak{f}_j\rangle\leqslant\varphi\langle\mathfrak{f}_j''\rangle$ and we must have equality in at least one instance. Therefore

$$\varphi\langle\mathfrak{f}_j\rangle = \varphi\langle\mathfrak{f}_j'\rangle = \gamma < \varphi\langle\mathfrak{f}_j''\rangle, \qquad 1\leqslant j\leqslant m \tag{10}$$

Now, for $i\neq j$ we find

$$\begin{aligned}2\varphi\langle\mathfrak{f}_i',\mathfrak{f}_j'\rangle &= 2\varphi\langle\mathfrak{f}_i-\mathfrak{f}_i'',\mathfrak{f}_j-\mathfrak{f}_j''\rangle \geqslant \min\{2\varphi\langle\mathfrak{f}_i,\mathfrak{f}_j''\rangle, 2\varphi\langle\mathfrak{f}_i'',\mathfrak{f}_j\rangle, 2\varphi\langle\mathfrak{f}_i'',\mathfrak{f}_j''\rangle\}\\ &\geqslant \min\{\varphi\langle\mathfrak{f}_i\rangle+\varphi\langle\mathfrak{f}_j''\rangle, \varphi\langle\mathfrak{f}_i''\rangle+\varphi\langle\mathfrak{f}_j\rangle, \varphi\langle\mathfrak{f}_i''\rangle+\varphi\langle\mathfrak{f}_j''\rangle\} > 2\gamma\end{aligned}$$

so that

(11) $$\varphi\langle \mathfrak{f}'_i, \mathfrak{f}'_j\rangle > \gamma, \quad 1 \leqslant i \neq j \leqslant m$$

Thus $\mathfrak{f}'_1, \ldots, \mathfrak{f}'_m$ are an almost orthogonal system in the $\nu(t)$-dimensional space $k(\mathfrak{e}_i)_{i\in J}$. Assume by way of contradiction that the $\mathfrak{f}'_j$ were linearly dependent, $\sum_1^m \mu_i \mathfrak{f}'_i = 0$ and not all $\mu_i = 0$. Thus, for each

$$r \in \{1, \ldots, m\}, 0 = \sum \mu_i \langle \mathfrak{f}'_i, \mathfrak{f}'_r\rangle$$

and so for each r

$$\varphi\langle \mathfrak{f}'_r\rangle + \varphi(\mu_r) = \varphi(-\sum_{j\neq r} \mu_j \langle \mathfrak{f}'_j, \mathfrak{f}'_r\rangle) \geqslant \min_{j\neq r} \{\varphi(\mu_j) + \varphi\langle \mathfrak{f}'_j, \mathfrak{f}'_r\rangle\}.$$

Therefore, by (10) and (11), $\varphi(\mu_r) > \min_{j\neq r} \{\varphi(\mu_j)\}$ which tells that there is no smallest $\varphi(\mu_r)$ at all, a contradiction. Therefore, $\mathfrak{f}'_1, \ldots, \mathfrak{f}'_m$ are linearly independent and so $m \leqslant \nu(t)$, QED. By Lemma 27 we thus obtain

COROLLARY 26. *The function* ν *that counts types on an orthogonal basis of* $\mathfrak{E}$ *is the same on all bases.*

VI.3. THE TYPE CONDITION. Let $\mathfrak{E}$ be the completion of a $\aleph_0$-dimensional space $\mathfrak{F}$ which is spanned by an orthogonal basis $(\mathfrak{e}_i)_{i\in\mathbf{N}}$ that satisfies the type condition (Def. 21).

LEMMA 27. *Let* $\mathscr{B} = (\mathfrak{u}_i)_{i\in\mathbf{N}}$ *be a maximal orthogonal family in* $\mathfrak{E}$. *Then* $\mathscr{B}$ *satisfies the type condition and* $\mathfrak{x} = \sum_{\mathbf{N}} \langle \mathfrak{x}, \mathfrak{u}_i\rangle \langle \mathfrak{u}_i\rangle^{-1}\mathfrak{u}_i$ *for all* $\mathfrak{x} \in \mathfrak{E}$. *In particular, the span of* $\mathscr{B}$ *is dense in* $\mathfrak{E}$.

Proof. The assertion on the type condition follows directly from Lemma 25. Let then $\mathfrak{x} \in \mathfrak{E}$.

$$\varphi\langle\langle \mathfrak{x}, \mathfrak{u}_i\rangle \langle \mathfrak{u}_i\rangle^{-1}\mathfrak{u}_i\rangle = 2\varphi\langle \mathfrak{x}, \mathfrak{u}_i\rangle - \varphi\langle \mathfrak{u}_i\rangle \geqslant \varphi\langle \mathfrak{x}\rangle + \varphi\langle \mathfrak{u}_i\rangle - \varphi\langle \mathfrak{u}_i\rangle = \varphi\langle \mathfrak{x}\rangle.$$

Thus the family of vectors $\langle \mathfrak{x}, \mathfrak{u}_i\rangle \langle \mathfrak{u}_i\rangle^{-1}\mathfrak{u}_i$ is bounded; in fact, it is a null sequence as $\mathscr{B}$ satisfies the type condition, hence it is summable as $\mathfrak{E}$ is complete. Put $\mathfrak{y} := \sum_{\mathbf{N}} \langle \mathfrak{x}, \mathfrak{u}_i\rangle \langle \mathfrak{u}_i\rangle^{-1}\mathfrak{u}_i$. We have $\langle \mathfrak{u}_i, \mathfrak{y}-\mathfrak{x}\rangle = \langle \mathfrak{u}_i, \mathfrak{y}\rangle - \langle \mathfrak{u}_i, \mathfrak{x}\rangle = 0$, so $\mathfrak{x} - \mathfrak{y} = 0$ as $\mathscr{B}$ is a maximal orthogonal family. □

VII. The Main Theorem

We are now able to characterize the definite spaces whose topology is admissible (Def. 1). Refer to Definition 21 for "type condition".

THEOREM 28 [20]. *Let* $\mathfrak{E}$ *be a definite space in the sense of Definition 15. The following conditions are equivalent*

(i) $L_c(\mathfrak{E}) = L_s(\mathfrak{E})$ (cf. (1), (2), (3))

(ii) $L_c(\mathfrak{E}) = L_{\perp\perp}(\mathfrak{E})$ ("*the topology is admissible*", *Def. 1*)

(iii) k *is complete and* $\mathfrak{E}$ *is the completion of a* $\aleph_0$*-dimensional space spanned by an orthogonal basis that satisfies the type condition.*

Proof. (i) $\Rightarrow$ (ii) holds trivially because $L_s \subseteq L_{\perp\perp} \subseteq L_c$ by continuity of the form; (ii) $\Rightarrow$ (iii) was carried out in Chapter V. Just as in [18] we can establish (iii) $\Rightarrow$ (i). Let $\mathfrak{U} \in L_c(\mathfrak{E})$. Pick a maximal orthogonal family $(\mathfrak{v}_i)_{i \in I}$ in $\mathfrak{U}$ and extend it to a maximal orthogonal family $(\mathfrak{v}_i)_{I \cup J}$ in $\mathfrak{E}$. For $\mathfrak{x} \in \mathfrak{E}$ we have by Lemma 27 $\mathfrak{x} = \mathfrak{x}' + \mathfrak{x}''$ where $\mathfrak{x}' = \sum_I \langle \mathfrak{x}, \mathfrak{v}_i \rangle \langle \mathfrak{v}_i \rangle^{-1} \mathfrak{v}_i$ and $\mathfrak{x}'' = \sum_J \langle \mathfrak{x}, \mathfrak{v}_i \rangle \langle \mathfrak{v}_i \rangle^{-1} \mathfrak{v}_i$. Now $\mathfrak{x}' \in \overline{\mathfrak{U}} = \mathfrak{U}$. All that remains to be shown is $\mathfrak{x}'' \in \mathfrak{U}^{\perp}$. Now $\mathfrak{U}^{\perp}$ is closed so it suffices to show that $\mathfrak{v}_i \in \mathfrak{U}^{\perp}$ for all $i \in J$. To this end pick $\mathfrak{u} \in \mathfrak{U}$ and decompose $\mathfrak{u} = \mathfrak{u}' + \mathfrak{u}''$ (analogous to the decomposition of $\mathfrak{x}$): $\mathfrak{u}'' = \mathfrak{u} - \mathfrak{u}' \in \mathfrak{U} - \mathfrak{U} = \mathfrak{U}$. Now $\langle \mathfrak{u}'', \mathfrak{v}_i \rangle = 0$ for all $i \in I$ so $\mathfrak{u}'' = 0$ since $(\mathfrak{v}_i)_{i \in I}$ is a maximal orthogonal family. From

$$0 = \mathfrak{u}'' := \sum_J \langle \mathfrak{u}, \mathfrak{v}_i \rangle \langle \mathfrak{v}_i \rangle^{-1} \mathfrak{v}_i$$

we obtain $\langle \mathfrak{u}, \mathfrak{v}_i \rangle = 0$ $(i \in J)$. As $\mathfrak{u} \in \mathfrak{U}$ was arbitrary this says that $\mathfrak{v}_i \in \mathfrak{U}^{\perp}$ $(i \in J)$.

Q.E.D.

Remark 29. Let the definite space $\mathfrak{E}$ be the completion of $\mathfrak{F} = k(\mathfrak{e}_i)_{i \in \mathbf{N}}$, $(\mathfrak{e}_i)_{\mathbf{N}}$ an orthogonal family (that does not necessarily satisfy the type condition). If k is complete then $\mathfrak{E}$ is isometric to the k-space $\hat{\mathfrak{F}}$ of all sequences $(\lambda_i)_{i \in \mathbf{N}} \in k^{\mathbf{N}}$ such that $\lim_{\mathbf{N}}(2\varphi\lambda_i + \varphi\langle \mathfrak{e}_i \rangle) = \infty$ and equipped with the form $\langle (\lambda_i), (\mu_i) \rangle = \sum_{\mathbf{N}} \lambda_i \mu_i \langle \mathfrak{e}_i \rangle$. Indeed, the set $\hat{\mathfrak{F}}$ is a definite k-space and the map $\Psi : (\lambda_i) \to \sum \lambda_i e_i$ is a well defined isometry $\hat{\mathfrak{F}} \to \Psi(\hat{\mathfrak{F}}) \subset \mathfrak{E}$. By the "infinite Pythagoras" we have $\ker \Psi = 0$; on the other hand, Lemma 16 shows that Ψ is also surjective.

Thus *all* definite spaces that carry an admissible topology are (by Theorem 28) of the kind invented by Keller.

Remark 30. By Theorem 28 the isometry type of a definite space with admissible topology is characterized by the sequence $(\langle \mathfrak{e}_i \rangle)_{i \in \mathbf{N}}$ where $(\mathfrak{e}_i)_{i \in \mathbf{N}}$ is a maximal orthogonal family in $\mathfrak{E}$. Conversely, for each $(\alpha_i) \in k^{\mathbf{N}}$ there is a definite space $\mathfrak{E}$ with $L_c(\mathfrak{E}) = L_s(\mathfrak{E})$ admitting a maximal orthogonal family $(\mathfrak{e}_i)_{i \in \mathbf{N}}$ with $\langle \mathfrak{e}_i \rangle = \alpha_i$ ($i \in \mathbf{N}$) provided that

(A) $\xi_i := \varphi \alpha_i \in \Gamma$ satisfies the (type-) condition expressed in (8)

(B) The form $\langle\ ,\ \rangle$ defined on $\mathfrak{F} := k(\mathfrak{e}_i)_{i \in \mathbf{N}}$ by $\langle \mathfrak{e}_i, \mathfrak{e}_j \rangle = 0$ $(i \neq j)$, $\langle \mathfrak{e}_i \rangle = \alpha_i$ ($i \in \mathbf{N}$) is definite.

These two conditions are implemented by many fields. In order to satisfy (A) one may, e.g. pick fields of generalized formal power series that are complete under a valuation φ with group Γ a prescribed Hahn product [30, p. 31] with sufficiently many factors not 2-divisible, e.g. $\Gamma = \mathbf{Z}^{(\mathbf{N})}$ ordered antilexicographically. Let k be any field with (A) and $t \in \Gamma/2\Gamma$; set $\mathfrak{F}_t = \{\text{span } \mathfrak{e}_i \mid \varphi\alpha_i + 2\Gamma = t\}$. By (A) $\dim \mathfrak{F}_t < \infty$; furthermore

$$\mathfrak{F} = \oplus^{\perp} \{\mathfrak{F}_t \mid t \in \Gamma/2\Gamma\}.$$

In order to check whether the form $\langle\ ,\ \rangle$ satisfies the triangle inequality on $\mathfrak{F}$ it suffices to verify said inequality on each $\mathfrak{F}_t$. A. Fässler has given a handy criterium for $\langle\ ,\ \rangle$ to be definite if Hahnproducts Γ are used, as indicated, to construct k with (A), [6, Lemma 15, 16].

VIII. Appendix: Extending the Main Theorem to the class $\mathscr{E}$ of norm-topological spaces

The arguments applied to the spaces in the class $\mathscr{D}$ can be extended to a larger class $\mathscr{E}$. First we have (cf. Definition 15):

Definition 31. An infinite dimensional anisotropic quadratic space $(\mathfrak{E}; \langle\ ,\ \rangle)$ over a *-valued field $(k, *, \varphi, \Gamma)$ is called norm-topological if the sets $\mathfrak{U}_\gamma := \{\mathfrak{x} \in \mathfrak{E} \mid \varphi\langle \mathfrak{x} \rangle > \gamma\}$ form a 0-neighbourhood basis of a vector space topology on $\mathfrak{E}$. Let $\mathscr{E}$ be the class of all norm-topological spaces.

Definite spaces are norm-topological, obviously.

A proper subgroup Δ of Γ is *convex* (or isolated) if "$0 \leqslant x \leqslant y$ & $y \in \Delta$" implies "$x \in \Delta$". If the subgroup $\Delta \subset \Gamma$ is convex then the factor group Γ/Δ is ordered by setting $\gamma + \Delta \leqslant \delta + \Delta$ iff $\gamma < \delta$ or $\gamma - \delta \in \Delta$; furthermore, $\varphi_\Delta : k \to \Gamma/\Delta \cup \{\infty\}$ defined by $\varphi_\Delta(\alpha) = \varphi(\alpha) + \Delta$ is a valuation (a "coarser valuation") which yields the same topology on k as φ.

In order to make the mechanism of types work in the context of norm-topological spaces, i.e., in order to salvage the statement of Corollary 26 in the new context, the concept of type has to be coarsened as follows. For $\gamma \in \Gamma$ we introduce

(12) $$\Delta(\gamma) := \{\delta \in \Gamma \mid \forall n \in \mathbf{N} : n \mid \delta \mid \leqslant \mid \gamma \mid\}$$

and

(13) $$\Theta(\gamma) := \bigcap_{\delta \in \Gamma} \Delta(\gamma + 2\delta)$$

If $\gamma \neq 0$ then $\Delta(\gamma)$ is the largest convex subgroup of Γ not containing γ ([21]).

Remark 32. The group defined in (13) for $\gamma = \varphi\langle \mathfrak{e} \rangle$, $\mathfrak{e} \in \mathfrak{E}$, represents yet another possibility to introduce a "type" for the vectors in a definite space. The fundamental property expressed in Lemma 25 can be replaced and reproved (along the same lines), cf. [21]:

(14) If U is a convex subgroup in Γ and $(\mathfrak{e}_i)_{\mathbf{N}}$, $(\mathfrak{f}_i)_{\mathbf{N}}$ are two maximal orthogonal families in a norm-topological space that satisfies (iii) in Theorem 28 then

$$\operatorname{card}\{i \in I \mid \Theta(\varphi\langle \mathfrak{e}_i \rangle \underset{\neq}{\subset} U\} = \operatorname{card}\{j \in \mathbf{N} \mid \Theta(\varphi\langle \mathfrak{f}_j \rangle) \underset{\neq}{\subset} U\}.$$

One has the following analogue of Lemma 14:

LEMMA 33. ([21]). *Let* $(\mathfrak{E}; \langle\ ,\ \rangle; \varphi, \Gamma, *)$ *be a norm-topological space and* $\varphi(2) = 0$ *(cf. Remark 35 below). Then there is a valuation* $\tilde{\varphi} : k \to \tilde{\Gamma} \cup \{\infty\}$ *coarser than* φ *such that the following holds: Either* $(\mathfrak{E}; \langle\ ,\ \rangle; \tilde{\varphi}, \tilde{\Gamma}, *)$ *is a definite space, in the sense of Definition 15, or else there are no analytically nilpotent elements* $\alpha \in k$ *(i.e., for no* $\alpha \neq 0$ *shall we have* $\lim_{\mathbf{N}} \alpha^n = 0$) *and then the following weakened versions of the statements in Lemma 14 hold:*

(i)′ $\tilde{\varphi}_\Delta\langle \mathfrak{x} + \mathfrak{y} \rangle \geqslant \min\{\tilde{\varphi}_\Delta\langle \mathfrak{x} \rangle, \tilde{\varphi}_\Delta\langle \mathfrak{y} \rangle\}$

(ii)′ $\tilde{\varphi}\langle \mathfrak{x} \rangle \leqslant \tilde{\varphi}\langle \mathfrak{y} \rangle \;\&\; \langle \mathfrak{x}, \mathfrak{y} \rangle = 0 \Rightarrow \tilde{\varphi}_\Lambda\langle \mathfrak{x} \rangle = \tilde{\varphi}_\Lambda\langle \mathfrak{x} + \mathfrak{y} \rangle$

(iii)′ $\tilde{\varphi}\langle \mathfrak{x}, \mathfrak{y} \rangle \geqslant \min\{\tilde{\varphi}_\Delta\langle \mathfrak{x} \rangle, \tilde{\varphi}_\Delta\langle \mathfrak{y} \rangle\}$

(iv)′ $2\tilde{\varphi}_\Delta\langle \mathfrak{x}, \mathfrak{y} \rangle \geqslant \tilde{\varphi}_\Delta\langle \mathfrak{x} \rangle + \tilde{\varphi}_\Delta\langle \mathfrak{y} \rangle$
where $\Lambda = \Theta(\tilde{\varphi}\langle \mathfrak{x} \rangle)$ *and* $\Delta = \Theta(\tilde{\varphi}\langle \mathfrak{x} \rangle) \cap \Theta(\tilde{\varphi}\langle \mathfrak{y} \rangle)$.

The inequalities in Lemma 33 suffice to salvage all results proved previously on definite spaces; in particular we have the following strengthening of Theorem 28 (cf. Remark 35 below):

THEOREM 34 [21]. *Let* $\mathfrak{E}$ *be a norm-topological space in the sense of Definition 31 and assume* $\varphi(2) = 0$. *Then the statements (i), (ii), (iii) in Theorem 28 are equivalent.*

Remark 35. In Definition 15, Lemma 33 and in Theorem 34 we stipulated that $\varphi(2) = 0$ for the valuation φ of the base field. However, it is neither necessary to assume this nor that char k be different from two. As technicalities increase if 2 is not a unit for φ the general case has been banned from this elementary survey. Refer to [21].

IX. APPENDIX: ORTHOMODULAR SPACES OVER ORDERED FIELDS

A Baer order of a *-field k is a subset $\Pi \subset S := \{\alpha \in k \mid \alpha = \alpha^*\}$ *with* $1 \in \Pi$, $0 \notin \Pi$, $\Pi + \Pi \subset \Pi$, $\forall \alpha \neq 0: \alpha\Pi\alpha^* \subset \Pi$, $-\Pi \,\dot{\cup}\, \Pi = S\backslash\{0\}$. ([14]). The map $\alpha \mapsto \alpha^*\alpha =: \|\alpha\|$ has the properties of a norm and defines a topology on k; if * is continuous then k is a topological *-field [14, Theorem 4.1, p. 231]. The theory of positive definite orthomodular spaces over archimedean ordered fields is settled in [9]: There are but the classical Hilbert spaces over **R**, **C**, **H**. If the order is non-archimedean we shall assume that

(15) the subgroup S generated by all $\alpha^*\alpha^{-1}$ is bounded.

There is [14, Sec. 4.5, p. 234] a valuation on k that induces the norm-topology. We remark that the boundness condition on S is always satisfied for the usual orderings on commutative fields, for Prestel's semi-orderings and for all *-ordered fields that are known hitherto.

A family $(e_\iota)_{\iota \in I}$ of vectors in a *positive definite space* $(\mathfrak{E}; \langle\ ,\ \rangle)$ over an ordered *-field k is said to satisfy the *type condition* (cf. Definition 21) iff for all $(\alpha_\iota)_{\iota \in I} \in k^I$ the following holds: if $(\langle \alpha_\iota e_\iota \rangle)_{\iota \in I}$ is bounded then $(\alpha_\iota e_\iota)_{\iota \in I}$ converges to $0 \in \mathfrak{E}$.

With this version of type condition we have

THEOREM 36. *Let* $(\mathfrak{E}; \langle\ ,\ \rangle)$ *be a positive definite space over a non-archimedean ordered *-field that satisfies (15). Then the statements (i), (ii), (iii) in Theorem 28 are equivalent.*

X. Clifford algebras of orthomodular spaces

X.1. Assumptions. In Chap. X k is a commutative field of characteristic not 2 and $\langle\ ,\ \rangle$ is a symmetric bilinear anisotropic form $\mathfrak{E} \times \mathfrak{E} \to k$ on the k-vector space $\mathfrak{E}$.

$C(\mathfrak{E})$ is the Clifford algebra of $(\mathfrak{E}; \langle\ ,\ \rangle)$; it is a k-algebra that contains the space $\mathfrak{E}$ as a set of ring generators which satisfy $\mathfrak{x} \cdot \mathfrak{y} + \mathfrak{y} \cdot \mathfrak{x} = 2\langle \mathfrak{x}, \mathfrak{y}\rangle$. For any pair of elements $\mathfrak{c}, \mathfrak{d} \in C(\mathfrak{E})$ there exists a finite orthogonal family $\mathfrak{e}_0, \ldots, \mathfrak{e}_n$ in $\mathfrak{E}$ such that $\mathfrak{c} = \sum\limits_I \alpha_I \mathfrak{e}_I$, $\mathfrak{d} = \sum \beta_I \mathfrak{e}_I$; here the summation index I runs over all subsets

$I = \{\iota_1 < \ldots < \iota_r\}$ of $\{0, 1, \ldots, n\}$ and $\mathfrak{e}_I := \mathfrak{e}_{\iota_1} \cdot \mathfrak{e}_{\iota_2} \cdot \ldots \cdot \mathfrak{e}_{\iota_r}$; the empty product $\mathfrak{e}_\varnothing$ is the unit element in $C(\mathfrak{E})$.

There is a *canonical* symmetric bilinear form $\langle\ ,\ \rangle$ on $C(\mathfrak{E})$ which extends the given form on $\mathfrak{E}$ ([5, 11, 22]). One has

$$\langle \mathfrak{c}, \mathfrak{d} \rangle = \sum_I \alpha_I \beta_I \prod_{\iota \in I} \langle \mathfrak{e}_\iota, \mathfrak{e}_\iota \rangle \tag{16}$$

From now on we shall assume that $(\mathfrak{E}; \langle\ ,\ \rangle)$ is an infinite dimensional definite space.

X.2. Clifford algebras of definite spaces. In [6] Angela Fässler has proved that for certain definite orthomodular spaces $\mathfrak{E}$ the algebra $C(\mathfrak{E})$ is a skew field; furthermore, the k-vector space $C(\mathfrak{E})$ equipped with the form (16) is a definite space whose completion $\tilde{C}(\mathfrak{E})$ is orthomodular again. Furthermore $\tilde{C}(\mathfrak{E})$ is a skew field, in fact, a *-valued field with * the extension to $\tilde{C}(\mathfrak{E})$ of the main antiautomorphism of the Clifford algebra $C(\mathfrak{E})$; the residue class field of $\tilde{C}(\mathfrak{E})$ is isomorphic to the residue class field of φ.

In the following theorem we prove the main fact in a simplified and slightly more general setting.

Theorem 37. *Assume that in the definite space* $(\mathfrak{E}; \langle\ ,\ \rangle)$ *each orthogonal family* $\mathfrak{e}_0, \ldots \mathfrak{e}_n$ *has*

$$\varphi\langle \mathfrak{e}_0 \rangle + \ldots + \varphi\langle \mathfrak{e}_n \rangle \notin 2\Gamma \tag{17}$$

Then:

(i) $C(\mathfrak{E})$ *equipped with the form in (16) is a definite space,*

(ii) $C(\mathfrak{E})$ *is a division ring,*

(iii) *The map* $\tilde{\varphi}: C(\mathfrak{E}) \to \Gamma \cup \{\infty\}$ *defined by* $\mathfrak{c} \mapsto \varphi\langle\mathfrak{c}\rangle$ *is a* $*$*-valuation for* $*$ *the main antiautomorphism of* $C(\mathfrak{E})$.

Proof. (i) It suffices to prove the triangle inequality (Lemma 14 (i)). Write $\mathfrak{c} = \sum \alpha_I \mathfrak{e}_I$, $\mathfrak{d} = \sum \beta_I \mathfrak{e}_I$ as in X.1. Then we have $\varphi\langle\alpha\mathfrak{e}_I\rangle \neq \varphi\langle\beta\mathfrak{e}_J\rangle$ for $I \neq J$ and $\alpha \neq 0 \neq \beta$. Hence

$$\varphi\langle\mathfrak{c}\rangle = \varphi\sum\langle\alpha_I\mathfrak{e}_I\rangle = \min_I\{\varphi\langle\alpha_I\mathfrak{e}_I\rangle\}$$

and similarly for $\varphi\langle\mathfrak{d}\rangle$. Therefore

$$\begin{aligned}\varphi\langle\mathfrak{c}+\mathfrak{d}\rangle &= \varphi\sum\langle(\alpha_I+\beta_I)\mathfrak{e}_I\rangle \geqslant \min\{2\varphi(\alpha_I+\beta_I) + \varphi\langle\mathfrak{e}_I\rangle\}\\ &\geqslant \min\{2\varphi\alpha_I + \varphi\langle\mathfrak{e}_I\rangle, 2\varphi\beta_I + \varphi\langle\mathfrak{e}_I\rangle\} = \min\{\varphi\langle\mathfrak{c}\rangle, \varphi\langle\mathfrak{d}\rangle\}.\end{aligned}$$

This proves (i). Next we show

$$\varphi\langle\mathfrak{c}\cdot\mathfrak{d}\rangle = \varphi\langle\mathfrak{c}\rangle + \varphi\langle\mathfrak{d}\rangle \tag{18}$$

Indeed, from

$$\langle\mathfrak{e}_I\cdot\mathfrak{e}_J\rangle = \langle\pm\langle\mathfrak{e}_{I\cap J}\rangle\mathfrak{e}_{(I\cup J)\setminus(I\cap J)}\rangle = \langle\mathfrak{e}_{I\cap J}\rangle^2\langle\mathfrak{e}_{(I\cup J)\setminus(I\cap J)}\rangle = \langle\mathfrak{e}_I\rangle\cdot\langle\mathfrak{e}_J\rangle$$

we see that

$$\varphi\langle\alpha_{I'}\,\mathfrak{e}_{I'}\rangle \leqslant \varphi\langle\alpha_I\mathfrak{e}_I\rangle \;\&\; \varphi\langle\beta_{J'}\,\mathfrak{e}_{J'}\rangle \leqslant \varphi\langle\beta_J\mathfrak{e}_J\rangle$$

implies

$$\varphi\langle\alpha_I, \beta_J, \mathfrak{e}_I, \mathfrak{e}_{J'}\rangle \leqslant \varphi\langle\alpha_I\beta_J\mathfrak{e}_I\mathfrak{e}_J\rangle.$$

We therefore pick $G, H \subseteq \{0, ..., n\}$ such that for all $I \subset \{0, ..., n\}$ we shall have

$$\varphi\langle\alpha_G\mathfrak{e}_G\rangle \leqslant \varphi\langle\alpha_I\mathfrak{e}_I\rangle, \varphi\langle\beta_H\mathfrak{e}_H\rangle \leqslant \varphi\langle\beta_I\mathfrak{e}_I\rangle.$$

It now follows that

$$\begin{aligned}\varphi\langle\mathfrak{c}\cdot\mathfrak{d}\rangle &= \varphi\langle(\textstyle\sum\alpha_I\mathfrak{e}_I)\cdot\sum\beta_J\mathfrak{e}_J\rangle = \varphi\langle\sum\alpha_I\beta_J\mathfrak{e}_I\mathfrak{e}_J\rangle = \varphi\langle\alpha_G\beta_H\mathfrak{e}_G\mathfrak{e}_H\\ &\quad + \textstyle\sum'\alpha_I\beta_J\mathfrak{e}_I\mathfrak{e}_J\rangle = \varphi\langle\alpha_G\beta_H\mathfrak{e}_G\mathfrak{e}_H\rangle = \varphi\langle\mathfrak{c}\rangle + \varphi\langle\mathfrak{d}\rangle.\end{aligned}$$

Thus (18) is established.

From (18) it follows that $C(\mathfrak{E})$ has no zero divisors, hence $C(\mathfrak{E})$ is a division ring (being an inductive limit of finite dimensional algebras). The map $\tilde{\varphi}: C(\mathfrak{E}) \to \Gamma \cup \{\infty\}$ as defined in (iii) of the Theorem is a $*$-valuation, for $\tilde{\varphi}(\mathfrak{c}^*) = \tilde{\varphi}(\mathfrak{c})$ is obvious and everything else has been established already.

COROLLARY 38. *Assume that the definite space* $(\mathfrak{E}; \langle\ ,\ \rangle)$ *is complete and that the system of types (Corollary 26) is linearly independent in* $\Gamma/2\Gamma$ *(considered as a* $\mathbf{Z}_2$*-vector space) then the conclusions (i), (ii), (iii) of Theorem 37 hold.*

$C(\mathfrak{E})$ in Theorem 37 is not complete (unless finite dimensional). Its quadratic form $\langle\ ,\ \rangle$ can be extended to the completion $\tilde{C}$. By using Theorem 28 one can see that this completion has $L_{\perp\perp}(\tilde{C}) = L_c(\tilde{C})$ if and only if E has $L_{\perp\perp}(E) = L_c(E)$.

XI. CONTINUOUS OPERATORS ARE NOT ALWAYS BOUNDED

XI.1. INTRODUCTION. Let $\mathfrak{E}$ be an infinite dimensional definite space in the sense of Definition 15. A linear map (operator) $h: \mathfrak{E} \to \mathfrak{E}$ is called *bounded* iff there exists $\gamma \in \Gamma$ such that for all $\mathfrak{x} \in \mathfrak{E}$ we have $\varphi\langle h\mathfrak{x}\rangle \geqslant \gamma + \varphi\langle \mathfrak{x}\rangle$.

In [6] A. Fässler gave an explicit example of a continuous operator h on an orthomodular space $\mathfrak{E}$ that is not bounded; she also proved a criterion for boundness which is very useful in the study of the algebra $\mathscr{B}(\mathfrak{E})$ of bounded operators $h: \mathfrak{E} \to \mathfrak{E}$ when $\mathfrak{E}$ is an orthomodular definite space of a certain kind. We shall prove this criterion anew here as its original proof can be shortened considerably.

We shall consider definite spaces that satisfy

(19) $(\mathfrak{E}; \langle\ ,\ \rangle)$ contains a maximal orthogonal family $(\mathfrak{e}_i)_{\mathbf{N}}$ such that the groups $\Theta(\varphi\langle \mathfrak{e}_i\rangle)$ are different.

By (14) we see that (19) is a property of $\mathfrak{E}$, not of $(\mathfrak{e}_i)_{\mathbf{N}}$; Keller's original example of an orthomodular space satisfies (19).

XI.2. FÄSSLER'S CRITERION. In this subsection let $(\mathfrak{E}; \langle\ ,\ \rangle)$ be an infinite dimensional orthomodular space that has (19). Fix a maximal orthogonal family $(\mathfrak{e}_i)_{\mathbf{N}}$ that enjoys (19). If $f: \mathfrak{E} \to \mathfrak{E}$ is given, expand (Lemma 27)

(20) $$f\mathfrak{e}_i = \sum_{j \in \mathbf{N}} \alpha_{ij}\mathfrak{e}_j \qquad (i \in \mathbf{N})$$

THEOREM 39 ([6]). *The linear map* f *is bounded iff it is continuous and satisfies*

(21) $\{\varphi\alpha_{ii} \mid T\,\varphi\langle f\mathfrak{e}_i\rangle = T\,\varphi\langle \mathfrak{e}_i\rangle\}$ *is bounded below.*

The heart of the proof of Theorem 39 is the following consequence of assumption (19).

LEMMA 40 [6]. *If f is continuous then (19) implies that the set $I := \{i \in \mathbf{N} \mid \varphi\langle f\mathfrak{e}_i\rangle < \varphi\langle \mathfrak{e}_i\rangle$ & $\varphi\langle f\mathfrak{e}_i\rangle \not\equiv \varphi\langle \mathfrak{e}_i\rangle \pmod{2\Gamma}\}$ is finite.*

Proof. We renumber the $\mathfrak{e}_i$ such that $\Theta(\varphi\langle \mathfrak{e}_i\rangle) \underset{\neq}{\subset} \Theta(\varphi\langle \mathfrak{e}_{i+1}\rangle)$. If we replace $\mathfrak{e}_i$ by a multiple then its group does not change; therefore we may assume without loss of generality that for all $r, s \in \mathbf{N}$ we have

(22) $$r < s \Rightarrow \varphi\langle \mathfrak{e}_r\rangle \in \Theta(\varphi\langle \mathfrak{e}_s\rangle)\,, \quad \varphi\langle \mathfrak{e}_r\rangle \geqslant 0$$

From (22) we obtain that for all $r, s \in \mathbf{N}$

(23) $$r < s \Rightarrow \forall\delta \in \Gamma: \varphi\langle \mathfrak{e}_r\rangle < |\,\varphi\langle \mathfrak{e}_s\rangle + 2\delta\,|$$

If $i \in I$ then $\varphi\langle f\mathfrak{e}_i\rangle \equiv \varphi\langle \mathfrak{e}_j\rangle$ for some $j \neq i$. Let $I_0 \subset I$ be the subset of those i for which the j is smaller than i. Thus, if $i \in I\backslash I_0$ then $\varphi\langle f\mathfrak{e}_i\rangle = \varphi\langle \mathfrak{e}_j\rangle + 2\varphi\alpha_{ij} < \varphi\langle \mathfrak{e}_i\rangle$; so by (23) we must actually have $\varphi\langle f\mathfrak{e}_i\rangle \leqslant -\varphi\langle \mathfrak{e}_i\rangle \leqslant 0$. Since $(\mathfrak{e}_i)$ is a null sequence we see that $I\backslash I_0$ has to be finite (because $\{f\mathfrak{e}_i \mid i \in I\backslash I_0\}$ must also be a null sequence if $I\backslash I_0$ is infinite). Thus, in order to prove Lemma 40 we have to show that I_0 is finite.

The idea in [6] ist to show that for each $i \in I_0$ there is $\lambda_i \in k$ such that $\varphi\langle f(\lambda_i\mathfrak{e}_i)\rangle \leqslant 0$ and $\varphi\langle \lambda_i\mathfrak{e}_i\rangle \geqslant 0$ so that by the same token I_0 must be finite. This is accomplished by choosing, in turn, $\lambda = 1$, $\lambda = \langle f\mathfrak{e}_i\rangle^{-1}$, according as to whether $\varphi\langle f\mathfrak{e}_i\rangle$ is $\leqslant 0$, > 0 respectively.

Proof of Theorem 39. Assume that f is bounded. Continuity is obvious. Let $\gamma \in \Gamma$ be a bound for f and let $\gamma_0 = \min\{0, \gamma\}$. Now $\varphi\langle f\mathfrak{e}_i\rangle = \varphi\langle \alpha_{ii}\mathfrak{e}_i\rangle$ for all i ocurring in (21), i.e., for all $i \in \mathbf{N}\backslash I$ (by assumption (19) we have $T\varphi\langle \mathfrak{e}_i\rangle \neq T\varphi\langle \mathfrak{e}_j\rangle$ for all $i \neq j$). Thus, if $\varphi\alpha_{ii} > 0$ then trivially $\varphi\alpha_{ii} \geqslant \gamma_0$; if $\varphi\alpha_{ii} < 0$ then $\varphi(\alpha_{ii}) > 2\varphi\alpha_{ii} \geqslant \gamma \geqslant \gamma_0$.

Assume conversely that f is continuous and has (21). We show that there is $\gamma_0 \in \Gamma$ with $\varphi\langle f\mathfrak{e}_i\rangle \geqslant \gamma_0 + \varphi\langle \mathfrak{e}_i\rangle$ ($i\in\mathbf{N}$). Let γ be a lower bound for the set in (21) and set $\gamma_0 := \min\{0, 2\gamma, \gamma_1, \ldots, \gamma_n\}$ where $\gamma_\nu := \varphi\langle f\mathfrak{e}_\nu\rangle - \varphi\langle \mathfrak{e}_\nu\rangle$, $\nu \in I$. To finish the proof we conclude $\varphi\langle f\mathfrak{x}\rangle > \varphi\langle \mathfrak{x}\rangle + \gamma_0$ $(\forall\mathfrak{x})$ by continuity of f:

$$\varphi\langle f\sum_1^\infty \xi_i\mathfrak{e}_i\rangle = \varphi\langle f(\xi_{i_0}\mathfrak{e}_{i_0})\rangle \geqslant \gamma_0 + \varphi\langle \xi_{i_0}\mathfrak{e}_{i_0}\rangle = \gamma_0 + \varphi\langle \mathfrak{x}\rangle\,.$$

XII. The Closed Graph Theorem

Let $\mathfrak{E}$, $\mathfrak{F}$ be definite spaces in the sense of Definition 15 over a field k whose valuation topology satisfies the 1. axiom of countability. For $f:\mathfrak{E} \to \mathfrak{F}$ a linear map set $\mathfrak{G}(f) := \{(\mathfrak{x}, \mathfrak{y}) \in \mathfrak{E} \oplus \mathfrak{F} \mid \mathfrak{y} = f(\mathfrak{x})\}$. Then [21] the "closed graph theorem" can be proved by classical methods (Baire category arguments):

$$\mathfrak{G}(f) \text{ is closed} \Rightarrow f \text{ is continuous}. \tag{24}$$

There is the following algebraic analogue of statement (24):

$$\mathfrak{G}(f) = \mathfrak{G}(f)^{\perp\perp} \Rightarrow f \text{ is } \perp\text{-continuous} \tag{25}$$

Here $\mathfrak{G}(f)^{\perp\perp}$ is taken in $\mathfrak{E} \overset{\perp}{\oplus} \mathfrak{F}$ and, by definition, f is $\perp$-continuous iff f is continuous with respect to the topologies on $\mathfrak{E}$ and $\mathfrak{F}$ whose 0-neighbourhood filters are generated by the orthogonals of all *finite* dimensional subspaces of $\mathfrak{E}$ and $\mathfrak{F}$ respectively. For $\mathfrak{E}$ an orthomodular space implication (25) holds: $\mathfrak{G}(f) = \mathfrak{G}(f)^{\perp\perp}$ implies that $\mathfrak{G}(f)$ is closed since the form is continuous on $\mathfrak{E} \overset{\perp}{\oplus} \mathfrak{F}$; so f is continuous by (24). Further, if $\mathfrak{G} \subset \mathfrak{F}$ is the orthogonal of a finite dimensional subspace then $f^{-1}(\mathfrak{G})$ is closed, hence $f^{-1}(\mathfrak{G}) = (f^{-1}(\mathfrak{G}))^{\perp\perp}$ as $\mathfrak{E}$ is orthomodular. But $(f^{-1}(\mathfrak{G}))^{\perp}$ is finite dimensional, hence f is $\perp$-continuous.

In [31] nice examples of $f:\mathfrak{E} \to \mathfrak{F}$ are given which illustrate that (25) is in general violated.

XIII. A few open problems

All orthomodular spaces are meant to be infinite dimensional and different from the classical ones over **R**, **C**, **H**.

Problem 1. Are cardinalities of maximal orthogonal families in an orthomodular space always equal? The answer is "yes" for those in $\mathscr{E}$.

Problem 2. Give an example of an orthomodular space that contains an uncountable orthogonal family of non-zero vectors.

Problem 3. Does the implication

$$\mathfrak{A} + \mathfrak{B} = (\mathfrak{A}+\mathfrak{B})^{\perp\perp} \Rightarrow \mathfrak{A}^{\perp} + \mathfrak{B}^{\perp} = (\mathfrak{A}\cap\mathfrak{B})^{\perp}$$

hold for all pairs of $\perp$-closed subspaces $\mathfrak{A} = \mathfrak{A}^{\perp\perp}$, $\mathfrak{B} = \mathfrak{B}^{\perp\perp}$ in an orthomodular space? The answer is "yes" for orthomodular spaces in $\mathscr{E}$. Cf. Remark 3 in [31]. More generally, are there other elementary lattice theoretic statements (in the sense of first order logic) that are valid in all $L_{\perp\perp}(E)$ where $\mathfrak{E}$ is orthomodular?

Problem 4. Are there spaces $\mathfrak{E}$ in $\mathscr{D}$, $\mathscr{E}$ with $L_s(\mathfrak{E}) = L_{\perp\perp}(\mathfrak{E}) \underset{\neq}{\subset} L_c(\mathfrak{E})$?

Problem 5. An orthomodular space $\mathfrak{E}$ in $\mathscr{E}$ is *never* isometric to any of its proper subspaces $\mathfrak{X}$, although it does happen that $\mathfrak{E}$ is similar to a proper subspace $\mathfrak{X}$. However, Keller's space is not similar to any of its proper subspaces. Give an intrinsic description of the phenomenon. (See [21].)

Problem 6. Answer Keller's question in § 3 of the introduction: When is $\{A\}'$ commutative for selfadjoint A in the algebra $\mathscr{B}(\mathfrak{H})$ of bounded operators $\mathfrak{H} \to \mathfrak{H}$?

Problem 7. Let $\mathfrak{E}$ be an orthomodular space in $\mathscr{D}$ or $\mathscr{E}$ such that the types of the members of a maximal orthogonal family are all different. Let Λ be the (countable) set of these types. For each choice of a family $(\lambda_i)_{i\in\Lambda}$ of nonnegative real numbers with $\sum_{\Lambda} \lambda_i = 1$ there is a probability distribution $f: L_{\perp\perp}(\mathfrak{E}) \to [0, 1] \subset \mathbf{R}$ uniquely defined as follows: for $\mathfrak{X} \in L_{\perp\perp}(\mathfrak{E})$ set $f(\mathfrak{X}) := \sum_{i\in J} \lambda_i$ where the subset $J \subseteq \Lambda$ consists of the types of the members of any orthogonal basis of $\mathfrak{X}$. We have $f(\mathfrak{E}) = 1$, $f(0) = 0$, $f(\sum \mathfrak{X}_i) = \sum f(\mathfrak{X}_i)$ for any countable family $\mathfrak{X}_0, \mathfrak{X}_1, \ldots$ of mutually orthogonal ($\perp$-closed) subspaces. These are by no means all probability distributions on $\mathfrak{E}$. There is a host of other possibilities. Can one bring some order into this multitude?

Problem 8. Classify the definite spaces with admissible topology over fixed base field.

Problem 9. Study the orthogonal group of definite orthomodular spaces.

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NOTE (added in proof)

The following recent papers are intimately related to the topics discussed above:

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