I. Orthomodular spaces (Terminology)

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measuring devices in the laboratory are by necessity archimedean ordered is besides the point, for, scales are not connected with the division ring underlying the space $\mathfrak{H}$ but with the range **R** of the probability distributions

$$f : L_{\perp\perp}(\mathfrak{H}) \to [0, 1] \subset \mathbf{R}$$

that thrive on the lattice $L_{\perp\perp}(\mathfrak{H})$. Remarkably enough, there is a lavish supply of real valued probability distributions on $L_{\perp\perp}(\mathfrak{H})$ for our non-classical orthomodular spaces $\mathfrak{H}$ in spite of the teratological nature of the base fields (cf. Problem 7 in XIII). *Independent of any axiomatics there is the fascinating mathematical problem to classify these probability distributions.* No approach à la Gleason is possible here [8].

The present paper is meant as an introduction to the topic of orthomodular quadratic spaces. Attention is restricted to hermitean spaces $(\mathfrak{E}; \langle\ ,\ \rangle)$ over valued fields or ordered fields. Let $\mathscr{E}$ be the class of all spaces $\mathfrak{E}$ which admit a vector space topology that makes $\langle\ ,\ \rangle$ continuous (Section VIII). For expository purposes our main interest here is in the subclass $\mathscr{D} \subset \mathscr{E}$ of all "definite" spaces (Definition 15): these are the spaces $\mathfrak{E}$ where a norm defined on $\mathfrak{E}$ via the form $\langle\ ,\ \rangle$ and the valuation (ordering respectively) satisfies a Cauchy-Schwarz type inequality (Section IV). In both classes $\mathscr{D}$, $\mathscr{E}$ the spaces satisfying (P_1) can be characterized (Theorems 28, 34, 36); these spaces satisfy (P_2) as well. This characterization allows to construct orthomodular spaces at will.

We further give a survey of some older results related to orthomodular spaces (Section II). We also append a list of open problems.

I. Orthomodular spaces (Terminology)

I.1 Conventions for the whole paper: In this paper we consider left vector spaces $\mathfrak{E}$ over division rings k with involution $\alpha \mapsto \alpha^*$ (antiautomorphism of k whose square is the identity). $\mathfrak{E}$ is equipped with an anisotropic hermitean form $\langle\ ,\ \rangle$; thus by definition for all

$$\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in \mathfrak{E}, \alpha \in k:$$

$$\langle \alpha\mathfrak{a}+\mathfrak{b}, \mathfrak{c}\rangle = \alpha\langle \mathfrak{a}, \mathfrak{c}\rangle + \langle \mathfrak{b}, \mathfrak{c}\rangle, \langle \mathfrak{a}, \mathfrak{b}\rangle = \langle \mathfrak{b}, \mathfrak{a}\rangle^*, \langle \mathfrak{a}, \mathfrak{a}\rangle = 0 \quad \text{iff} \quad \mathfrak{a} = 0.$$

We shall often abreviate "$\langle \mathfrak{a}, \mathfrak{a}\rangle$" by "$\langle \mathfrak{a}\rangle$". If $\mathfrak{E}$ is infinite dimensional there are always subspaces $\mathfrak{F}$ that are properly contained in their biorthogonals $\mathfrak{F}^{\perp\perp} := (\mathfrak{F}^{\perp})^{\perp}$ [10; Lemma 3, p. 20]. Let $L(\mathfrak{E})$ be the set of all linear subspaces of $\mathfrak{E}$ and

(1) $$L_{\perp\perp}(\mathfrak{E}) := \{\mathfrak{F} \in L(\mathfrak{E}) \mid \mathfrak{F} = \mathfrak{F}^{\perp\perp}\}$$

We are interested in the set of splitting subspaces

(2) $$L_s(\mathfrak{E}) := \{\mathfrak{F} \in L(\mathfrak{E}) \mid \mathfrak{F} + \mathfrak{F}^{\perp} = \mathfrak{E}\}$$

Clearly $L_s(\mathfrak{E}) \subset L_{\perp\perp}(\mathfrak{E})$. A hermitean space $\mathfrak{E}$ is called *orthomodular* iff $L_s = L_{\perp\perp}$. In [6, 7, 9, 10, 18, 20; 31, 32] orthomodular spaces and forms were termed "hilbertian". However, "hilbertian form" already has a different meaning in the theory of normed algebras [5, Chap. XV.6] which actually causes equivocations. We have therefore yielded to the "orthomodular"-terminology.

In the following k is usually assumed to be a topological division ring and $\mathfrak{E}$ equipped with a vector space topology τ (which means that τ is compatible with the additive group of $\mathfrak{E}$ and scalar multiplication $k \times \mathfrak{E} \to \mathfrak{E}$ is continuous) such that the form $\langle\ ,\ \rangle$ on $\mathfrak{E}$ is (separately) continuous. We then consider the set of closed linear subspaces in $(\mathfrak{E}, \tau)$

(3) $$L_c(\mathfrak{E}) := \{\mathfrak{F} \in L(\mathfrak{E}) \mid \overline{\mathfrak{F}} = \mathfrak{F}\}$$

We have $L_{\perp\perp}(\mathfrak{E}) \subseteq L_c(\mathfrak{E})$ by continuity of the form.

Definition 1. The vector space topology τ on $\mathfrak{E}$ is admissible if and only if $L_{\perp\perp}(\mathfrak{E}) = L_c(\mathfrak{E})$.

Remark 2. All (infinite dimensional) orthomodular spaces $\mathfrak{E}$ discovered hitherto carry an admissible topology and this topology is needed to handle the space. Furthermore, all orthomodular spaces other than classical Hilbert space are separable in the sense that they contain *countable* families with $\perp$-dense span. This is quaint. No non-separable orthomodular space has been discovered so far. Cf. Remark 8.

I.2 APPENDIX ON LATTICES. These brief remarks are not needed in order to understand the rest of the paper; however they throw light on concepts and related problems.

A *lattice* L is a non-void partially ordered set such that

$$a \vee b := \sup\{a, b\}, \quad a \wedge b := \inf\{a, b\}$$

exist for all pairs (and hence all finite sets) of elements of L. If arbitrary sets of elements of L admit suprema and infima then L is called *complete.* We always assume that L has *universal bounds* 0 and 1. An element b is said to *cover* an element a, $a < \cdot\, b$, when $a < b$ and for no c we have

$a < c < b$; *atoms* are elements that cover 0. A lattice is *atomistic* when every non-zero element a is the supremum of all atoms $\leqslant a$. The following property is *the covering property*: "if p is an atom and $a \wedge p = 0$ then $a <\cdot\, a \vee p$. Both $L(\mathfrak{E})$ and $L_{\perp\perp}(\mathfrak{E})$ are lattices with respect to $\subseteq$ whereas $L_s(\mathfrak{E})$ is not, in general, a lattice (cf. [9]). In fact, $L(\mathfrak{E})$ and $L_{\perp\perp}(\mathfrak{E})$ are complete, atomistic and they enjoy the covering property.

An *orthocomplementation* $a \mapsto a^{\perp}$ on a lattice L is a decreasing involution with $a^{\perp} \vee a = 1$, $a^{\perp} \wedge a = 0$. It follows that $(a \vee b)^{\perp} = a^{\perp} \wedge b^{\perp}$. An orthocomplemented lattice L is called *orthomodular* if its elements satisfy [15, p. 780]

$$a \leqslant b \Rightarrow b = a \vee (b \wedge a^{\perp}) \tag{4}$$

In an orthomodular lattice L we call *compatible* two elements a, b if $b = (b \wedge a) \vee (b \wedge a^{\perp})$; this is the case iff the orthocomplemented lattice generated by a, b is distributive ([29, (2.25) p. 28]). If 0, 1 are the only elements compatible with all elements of L then L is called *irreducible.* A *propositional system* is a complete, orthomodular, atomistic lattice that enjoys the covering property.

The lattice $L_{\perp\perp}(\mathfrak{E})$ attached to a hermitean space is always orthocomplemented (recall that we assume the forms to be non-isotropic). If $\mathfrak{E}$ is orthomodular, then $L_{\perp\perp}(\mathfrak{E})$ is an orthomodular lattice, and conversely (hence the terminology). In fact, one easily verifies:

(5) If $L_s(\mathfrak{E}) = L_{\perp\perp}(\mathfrak{E})$ then $L_{\perp\perp}(\mathfrak{E})$ is an irreducible propositional system.

The following converse of (5) is essentially due to G. Birkhoff and J. v. Neumann [4, Appendix] and R. Baer [2, p. 302] (Cf. [10, p. 45], [23]).

THEOREM 3. *Let L be any irreducible propositional system of dimension $\geqslant 4$ (i.e. there is a chain $0 < a < b < c < d$ in L). Then L is $\perp$-isomorphic to the lattice $L_{\perp\perp}(\mathfrak{E})$ of some suitable orthomodular space $\mathfrak{E}$ over a suitable division ring k.*

This theorem explains the interest that the quantum logic approach to axiomatic quantum mechanics had taken in propositional systems: they lead towards the classical interpretation. The rub is that the division ring k need *not* be **R**, **C** or **H** as we *know* since Keller's example [18]. In order to arrive at the classical structures stronger axioms on the lattice have to be postulated such as, for example, in [12, 33]. The reader interested in this kind of foundational problems in physics is refered to [3, 12, 15, 29].

Orthomodular lattices that derive from orthomodular quadratic spaces make up only a fraction of abstract orthomodular lattices (refer to [13, 16, 17]). The orthomodular law (4) is exceedingly enigmatic even if attention is restricted to orthomodular quadratic spaces. The complexity of the orthomodular conundrum does not surprise us anymore.

II. Results on orthomodular spaces prior to Keller's discovery

II.1. Results without topological restrictions on $\mathfrak{E}$. We begin with a classic ([1]).

Theorem 4 (Amemiya-Araki-Piron). *Let k be one of $\mathbf{R}, \mathbf{C}, \mathbf{H}$ and $\mathfrak{E}$ an infinite-dimensional k-vector space equipped with a positive definite hermitean form $\langle\ ,\ \rangle$ (relative to the usual involution $*$ in k). Then $\mathfrak{E}$ is orthomodular iff $\mathfrak{E}$ is complete as a normed space*

$$(\|\mathfrak{x}\| := \langle \mathfrak{x}, \mathfrak{x}\rangle^{\frac{1}{2}}),$$

i.e. iff $\mathfrak{E}$ is a Hilbert space.

If, in the setting of Thm. 4, we pass to subfields of k then the same conclusion can be drawn although the proof is much more tricky [9]:

Theorem 5 (Gross-Keller). *Let k be an archimedean (Baer-)ordered $*$-field* ([14, p. 219]) *and $\mathfrak{E}$ an infinite dimensional k-vector space equipped with a positive definite hermitean form. Then the following are equivalent*

(i) *k is one of $\mathbf{R}, \mathbf{C}, \mathbf{H}$ and $\mathfrak{E}$ is a Hilbert space*

(ii) *$L_s(\mathfrak{E}) = L_{\perp\perp}(\mathfrak{E})$ i.e. $\mathfrak{E}$ is orthomodular*

(iii) *$L_c(\mathfrak{E}) = L_{\perp\perp}(\mathfrak{E})$ (c refers to the norm $\|\mathfrak{x}\| := \langle \mathfrak{x}, \mathfrak{x}\rangle^{\frac{1}{2}} \in k^{\frac{1}{2}}$)*

(iv) *$L_s(\mathfrak{E}) = L_{\perp\perp}(\mathfrak{E}) = L_c(\mathfrak{E})$.*

Remark 6. In [24] sequence spaces $\mathfrak{E} := \ell_2(k)$ for $k \subset \mathbf{H}$ are considered and equipped with hermitean maps (not forms) $\mathfrak{E} \times \mathfrak{E} \to \mathbf{H}$. Again, the lattice of $\perp$-closed subspaces in $\mathfrak{E}$ is orthomodular iff $k = \mathbf{R}, \mathbf{C}$, or $\mathbf{H}$.

Another attempt to chance upon new orthomodular forms is to replace the reals by the non-archimedian ordered field $*\mathbf{R}$, a non-standard model of $\mathbf{R}$. However [28]: