II. Results on orthomodular spaces prior to Keller's discovery

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Orthomodular lattices that derive from orthomodular quadratic spaces make up only a fraction of abstract orthomodular lattices (refer to [13, 16, 17]). The orthomodular law (4) is exceedingly enigmatic even if attention is restricted to orthomodular quadratic spaces. The complexity of the orthomodular conundrum does not surprise us anymore.

II. Results on orthomodular spaces prior to Keller's discovery

II.1. Results without topological restrictions on $\mathfrak{E}$. We begin with a classic ([1]).

Theorem 4 (Amemiya-Araki-Piron). *Let k be one of $\mathbf{R}, \mathbf{C}, \mathbf{H}$ and $\mathfrak{E}$ an infinite-dimensional k-vector space equipped with a positive definite hermitean form $\langle\ ,\ \rangle$ (relative to the usual involution $*$ in k). Then $\mathfrak{E}$ is orthomodular iff $\mathfrak{E}$ is complete as a normed space*

$$(\|\mathfrak{x}\| := \langle \mathfrak{x}, \mathfrak{x}\rangle^{\frac{1}{2}}),$$

i.e. iff $\mathfrak{E}$ is a Hilbert space.

If, in the setting of Thm. 4, we pass to subfields of k then the same conclusion can be drawn although the proof is much more tricky [9]:

Theorem 5 (Gross-Keller). *Let k be an archimedean (Baer-)ordered $*$-field* ([14, p. 219]) *and $\mathfrak{E}$ an infinite dimensional k-vector space equipped with a positive definite hermitean form. Then the following are equivalent*

(i) *k is one of $\mathbf{R}, \mathbf{C}, \mathbf{H}$ and $\mathfrak{E}$ is a Hilbert space*

(ii) $L_s(\mathfrak{E}) = L_{\perp\perp}(\mathfrak{E})$ *i.e. $\mathfrak{E}$ is orthomodular*

(iii) $L_c(\mathfrak{E}) = L_{\perp\perp}(\mathfrak{E})$ *(c refers to the norm $\|\mathfrak{x}\| := \langle \mathfrak{x}, \mathfrak{x}\rangle^{\frac{1}{2}} \in k^{\frac{1}{2}}$)*

(iv) $L_s(\mathfrak{E}) = L_{\perp\perp}(\mathfrak{E}) = L_c(\mathfrak{E})$.

Remark 6. In [24] sequence spaces $\mathfrak{E} := \ell_2(k)$ for $k \subset \mathbf{H}$ are considered and equipped with hermitean maps (not forms) $\mathfrak{E} \times \mathfrak{E} \to \mathbf{H}$. Again, the lattice of $\perp$-closed subspaces in $\mathfrak{E}$ is orthomodular iff $k = \mathbf{R}, \mathbf{C}$, or $\mathbf{H}$.

Another attempt to chance upon new orthomodular forms is to replace the reals by the non-archimedian ordered field $*\mathbf{R}$, a non-standard model of $\mathbf{R}$. However [28]:

THEOREM 7 (Morash). *The inner product on* $\mathfrak{H} = \ell_2(\mathbf{R})$ *induces a positive definite symmetric bilinear form* $*\mathfrak{H} \times *\mathfrak{H} \to *\mathbf{R}$; *here* $*\mathfrak{H}$ *is the set (linear* $*\mathbf{R}$*-space) of equivalence classes in* $\mathfrak{H}^{\mathbf{N}}$ *induced by the free ultra filter* U *on* $\mathbf{N}$ *used to define* $*\mathbf{R}$. *The lattice* $L_{\perp\perp}(*\mathfrak{H})$ *is complete but not orthomodular.*

Remark 8. In [28] it is also shown that the ultra filter construction applied to a product of lattices isomorphic to $L_{\perp\perp}(\ell_2(\mathbf{R}))$ leads to an orthomodular lattice that, alas, is not complete. This loss of completeness, incidentally, is *the* (only) obstacle on the way to an easy (ultrafilter construction + Theorem 3) existence proof for orthomodular spaces different from Hilbert space.

A rather general theorem is ([33]):

THEOREM 9 (Wilbur). *Let* $(k, *)$ *be commutative and such that for each* $*$*-symmetric element* $\lambda \in k$ *there is* $\alpha \in k$ *with* $\lambda = \pm \alpha\alpha^*$. *If* $\mathfrak{E}$ *is an orthomodular space over* k, $\dim \mathfrak{E}$ *infinite, then* $k = \mathbf{R}$ *or* $\mathbf{C}$ *with* $*$ *the identity or the usual conjugation, respectively (so* $\mathfrak{E}$ *is a Hilbert space).*

Remark 10. The formulation of Thm. 9 in [33] also admits skew $(k, *)$ with one additional assumption. However, by Dieudonné's Lemma ([10 p. 18]) $(k, *)$ must then be a quaternion algebra with $*$ the usual conjugation.

Wilbur's result is generalized to ordered $*$-fields in [14, § 6].

Hermitean spaces that are orthogonal sums of finite dimensional subspaces are called *diagonal*; subspaces of diagonal spaces are termed *prediagonal.* There is a full-fledged theory about prediagonal spaces of infinite dimensions. Deplorably, we have ([9]):

THEOREM 11 (Gross-Keller). *Let* $\dim \mathfrak{E} \geqslant \aleph_0$. *If* $\mathfrak{E}$ *is prediagonal then it is not orthomodular. Thus, in particular,* $\dim \mathfrak{E} > \aleph_0$ *if* $\mathfrak{E}$ *is orthomodular.*

Orthomodularity of a space $\mathfrak{E}$ has strange consequences for the base field of $\mathfrak{E}$. We just mention one of several [9, p. 15].

THEOREM 12 (Gross-Keller). *If* $\operatorname{card} k < 2^{\aleph_0}$ *then an infinite dimension al* k*-space* $\mathfrak{E}$ *cannot be orthomodular.*

II.2. A RESULT ON SPACES $\mathfrak{E}$ EQUIPPED WITH AN ADMISSIBLE TOPOLOGY. Certain well known classes of spaces $\mathfrak{E}$ that carry admissible topologies can be proved *not* to contain orthomodular specimen; we refer to [9]. Here we mention but one result ([9, p. 20]); it has been crucial on the road to Keller's discovery. The idea of its proof is used again in the proof of Theorem 17 below.

THEOREM 13 (Gross-Keller). *Let k be a non archimedean ordered field and equipped with its order topology; let $\langle\ ,\ \rangle$ be a definite symmetric form on the k-vector space $\mathfrak{E}$. Equip $\mathfrak{E}$ with the norm topology*

$$(\|\mathfrak{x}\| := \langle \mathfrak{x}, \mathfrak{x}\rangle^{\frac{1}{2}} \in k^{\frac{1}{2}}) .$$

Assume that $\mathfrak{E}$ contains at least one orthogonal family $(\mathfrak{e}_i)_{i\in\mathbf{N}}$ that is bounded, i.e. for suitable $\alpha, \beta \in k$

$$0 < \alpha \leqslant \langle \mathfrak{e}_i, \mathfrak{e}_i\rangle \leqslant \beta \quad (i\in\mathbf{N}) \tag{6}$$

Then $L_{\perp\perp}(\mathfrak{E}) \underset{\neq}{\subset} L_c(\mathfrak{E})$.

III. KELLER'S EXAMPLE

The authors of [9] lamented about the "irksome" condition (6) which, indeed, need not be satisfied (*loc. cit.*, p. 89). Keller finally noticed that (6) pointed at the very crux of the matter. He considered the transcendental extension $k_0 = \mathbf{Q}(X_i)_{i\in\mathbf{N}}$ with the unique ordering that has $X_0 > q$ for all $q \in \mathbf{Q}$ and $X_i^n < X_{i+1}$ for all i and all n; then he let k be the completion of k_0 by means of Cauchy sequences. $\mathfrak{E}$ is the linear k-space of all $(y_i)_{i\in\mathbf{N}} \in k^{\mathbf{N}}$ such that $\sum_{\mathbf{N}} y_i^2 X_i$ exists (addition and scalar multiplication component wise) and $\langle (y_i)_{i\in\mathbf{N}}, (z_i)_{i\in\mathbf{N}}\rangle := \sum_{\mathbf{N}} y_i z_i X_i$. Original and ingenious arguments given in [18] establish orthomodularity of $\mathfrak{E}$. (This also follows from our Theorem 36 below.)

Gross noticed that Keller's construction works for valued fields ([6, 7, 20]). An example is also contained in [14, p. 237]).

Keller's choice of a field over which one can build orthomodular spaces has been good: as our results show his space exhibits the typical properties of an orthomodular space with an admissible topology (cf. Remark 29 below).