IV. The fundamental inequalities in definite spaces

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IV. THE FUNDAMENTAL INEQUALITIES IN DEFINITE SPACES

IV.1. *-VALUATIONS (cf. [14]). Let $(k, {}^*)$ be an involutorial division ring and Γ a totally ordered (additively written) abelian group. A surjective map

$$\varphi : k \to \Gamma \cup \{\infty\} \qquad (a + \infty = \infty \quad \text{for all} \quad a \in \Gamma \cup \{\infty\}) \tag{7}$$

is called *-valuation iff (i) $\varphi(x+y) \geqslant \min\{\varphi(x), \varphi(y)\}$, (ii) $\varphi(xy) = \varphi(x) + \varphi(y)$, (iii) $\varphi(x) = \infty \Leftrightarrow x = 0$, (iv) $\varphi(x) = \varphi(x^*)$.

The set of all $U_\varepsilon := \{x \in k \mid \varphi(x) \geqslant \varepsilon\}$, $\varepsilon \in \Gamma$, is a neighbourhood basis for a division ring topology on k. In general we think of $(k, {}^*)$ as equipped with this topology.

IV.2. THE INEQUALITIES. Assume that char $k \neq 2$ and that the valuation in (7) has $\varphi(2) = 0$ (cf. Remark 35). Let $\langle\ ,\ \rangle$ be a hermitean form on a k-space $\mathfrak{E}$. Assume $\mathfrak{E}$ non-degenerate ($\mathfrak{E}^\perp = (0)$). Recall that we write "$\langle \mathfrak{x} \rangle$" for $\langle \mathfrak{x}, \mathfrak{x} \rangle$, $\mathfrak{x} \in \mathfrak{E}$. It is useful to know a proof for the following fact

LEMMA 14 ([20]). *The following four statements are equivalent*

(i) $\forall \mathfrak{x}, \mathfrak{y} \in \mathfrak{E}: \varphi\langle \mathfrak{x} + \mathfrak{y} \rangle \geqslant \min\{\varphi\langle \mathfrak{x} \rangle, \varphi\langle \mathfrak{y} \rangle\}$ *(triangle inequality)*

(ii) $\forall \mathfrak{x}, \mathfrak{y} \in \mathfrak{E}: \langle \mathfrak{x}, \mathfrak{y} \rangle = 0 \Rightarrow \varphi\langle \mathfrak{x} + \mathfrak{y} \rangle = \min\{\varphi\langle \mathfrak{x} \rangle, \varphi\langle \mathfrak{y} \rangle\}$ *("Pythagoras")*

(iii) $\forall \mathfrak{x}, \mathfrak{y} \in \mathfrak{E}: \varphi\langle \mathfrak{x}, \mathfrak{y} \rangle \geqslant \min\{\varphi\langle \mathfrak{x} \rangle, \varphi\langle \mathfrak{y} \rangle\}$ *("weak Cauchy-Schwarz")*

(iv) $\forall \mathfrak{x}, \mathfrak{y} \in \mathfrak{E}: 2\varphi\langle \mathfrak{x}, \mathfrak{y} \rangle \geqslant \varphi\langle \mathfrak{x} \rangle + \varphi\langle \mathfrak{y} \rangle$ *("Cauchy-Schwarz")*

(Notice that each statement implies anisotropy of $\mathfrak{E}$*).*

Proof. (i) $\Rightarrow$ (ii): Let $\mathfrak{x} \perp \mathfrak{y}$ and

$$\varphi\langle \mathfrak{x} \rangle \leqslant \varphi\langle \mathfrak{y} \rangle; \quad \varphi\langle \mathfrak{x} \rangle = \varphi\langle 2\mathfrak{x} \rangle = \varphi\langle (\mathfrak{x}+\mathfrak{y}) + (\mathfrak{x}-\mathfrak{y}) \rangle \geqslant \min\{\varphi\langle \mathfrak{x}+\mathfrak{y} \rangle, \varphi\langle \mathfrak{x}-\mathfrak{y} \rangle\} = \varphi\langle \mathfrak{x}+\mathfrak{y} \rangle \geqslant \varphi\langle \mathfrak{x} \rangle.$$

(ii) $\Rightarrow$ (iv): Assume $\mathfrak{x} \neq 0 \neq \mathfrak{y}$. Pick $\mathfrak{b}$ in the span of $\mathfrak{x}, \mathfrak{y}$ such that

$$\mathfrak{x} = \mathfrak{b} + \lambda \mathfrak{y}, \ \mathfrak{b} \perp \mathfrak{y}; \quad 2\varphi\langle \mathfrak{x}, \mathfrak{y} \rangle = 2\varphi\langle \mathfrak{b} + \lambda\mathfrak{y}, \mathfrak{y} \rangle = 2\varphi\langle \lambda\mathfrak{y}, \mathfrak{y} \rangle$$
$$= 2\varphi(\lambda) + 2\varphi\langle \mathfrak{y} \rangle = \varphi\langle \lambda \mathfrak{y} \rangle + \varphi\langle \mathfrak{y} \rangle \geqslant \varphi\langle \mathfrak{x} \rangle + \varphi\langle \mathfrak{y} \rangle.$$

(iv) $\Rightarrow$ (iii): trivial

(iii) $\Rightarrow$ (i): straight forward. $\square$

IV.3. THE CLASS $\mathscr{D}$ OF DEFINITE SPACES. Positive definite forms over ordered fields satisfy the triangle inequality as well as the Cauchy-Schwarz inequality. We therefore set down

Definition 15. A definite space is a nondegenerate hermitean space $(\mathfrak{E}; \langle\ ,\ \rangle)$ over an involutorial division ring $(k, *)$, char $k \neq 2$, that is equipped with a $*$-valuation φ that has $\varphi(2) = 0$ (cf. Remark 35) and that satisfies one (and hence all) of the four statements in Lemma 14. A definite space $\mathfrak{E}$ will always be considered as a topological vector space, the topology being given by the zero-neighbourhood basis $\mathfrak{U}_\gamma := \{\mathfrak{y} \in \mathfrak{E} \mid \varphi\langle\mathfrak{y}\rangle \geqslant \gamma\}, \gamma \in \Gamma$. If $(\mathfrak{e}_\iota)_{\iota\in I}$ is any family over vectors in $\mathfrak{E}$ such that the net of all *finite* ("partial") sums $\sum \mathfrak{e}_\iota$ has a limit $\mathfrak{x}$ in $\mathfrak{E}$ then we write $\mathfrak{x} = \sum_{\iota\in I} \mathfrak{e}_\iota$ and call $(\mathfrak{e}_\iota)_{\iota\in I}$ summable.

LEMMA 16. *Let* $(\mathfrak{e}_\iota)_{\iota\in I}$ *be an orthogonal family in the definite space* $(\mathfrak{E}; \langle\ ,\ \rangle)$ *and* $\mathfrak{F}$ *its span. For each* $\mathfrak{x}$ *in the topological closure of* $\mathfrak{F}$ *we have* $\mathfrak{x} = \sum_{\iota\in I} \langle \mathfrak{x}, \mathfrak{e}_\iota\rangle \langle \mathfrak{e}_\iota\rangle^{-1}\mathfrak{e}_\iota$.

Proof. Let $\mathscr{P}$ be the set of all finite subsets of I. For $V \in \mathscr{P}$ we set $\mathfrak{x}_V := \sum_{\iota\in V} \langle \mathfrak{x}, \mathfrak{e}_\iota\rangle \langle \mathfrak{e}_\iota\rangle^{-1}\mathfrak{e}_\iota$. We have to prove that for each $\gamma \in \Gamma$ there is $U \in \mathscr{P}$ such that $\varphi\langle\mathfrak{x}-\mathfrak{x}_V\rangle \geqslant \varepsilon$ for all V *with* $U \subset V \in \mathscr{P}$. Now there is $\mathfrak{y} \in \mathfrak{F}$ with $\varphi(\mathfrak{x}-\mathfrak{y}\rangle \geqslant \varepsilon$. Pick $U \in \mathscr{P}$ with $\mathfrak{y} \in \operatorname{span}\{\mathfrak{e}_\iota \mid \iota \in U\}$. If $U \subset V \in \mathscr{P}$ then $\mathfrak{x} - \mathfrak{x}_V \perp \mathfrak{x}_V - \mathfrak{y}$, so by "Pythagoras" (Lemma 14 (ii)) we obtain $\varepsilon \leqslant \varphi\langle\mathfrak{x}-\mathfrak{y}\rangle = \min\{\varphi\langle\mathfrak{x}-\mathfrak{x}_V\rangle, \varphi\langle\mathfrak{x}_V-\mathfrak{y}\rangle\} \leqslant \varphi\langle\mathfrak{x}-\mathfrak{x}_V\rangle$. □

V. NECESSARY CONDITIONS IN $\mathscr{D}$ FOR $L_c = L_{\perp\perp}$

The principal result of this section is

THEOREM 17 ([20]). *Let* $\mathfrak{E}$ *be an infinite dimensional definite space carrying an admissible topology i.e., the topology mentioned in Definition 15 is admissible in the sense of Definition 1; let furthermore* $(\mathfrak{e}_\iota)_{\iota\in I}$ *be an orthogonal family in* $\mathfrak{E}$ *such that* $(\varphi\langle\mathfrak{e}_\iota\rangle)_{\iota\in I}$ *has a lower bound in* Γ. *Then* $\sum_{\iota\in I} \mathfrak{e}_\iota$ *exists.*

Proof. Let $\mathfrak{F} := \operatorname{span}\{\langle\mathfrak{e}_\iota\rangle^{-1}\mathfrak{e}_\iota - \langle\mathfrak{e}_0\rangle^{-1}\mathfrak{e}_0 \mid \iota \in I\}$. We first wish to show that $\langle\mathfrak{e}_0\rangle^{-1}\mathfrak{e}_0$ is not an element of the topological closure $\overline{\mathfrak{F}}$. Indeed,