V. Necessary conditions in D for $L_c = L_{\bot \bot}$

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IV.3. THE CLASS $\mathscr{D}$ OF DEFINITE SPACES. Positive definite forms over ordered fields satisfy the triangle inequality as well as the Cauchy-Schwarz inequality. We therefore set down

Definition 15. A definite space is a nondegenerate hermitean space $(\mathfrak{E}; \langle\ ,\ \rangle)$ over an involutorial division ring $(k, *)$, char $k \neq 2$, that is equipped with a $*$-valuation φ that has $\varphi(2) = 0$ (cf. Remark 35) and that satisfies one (and hence all) of the four statements in Lemma 14. A definite space $\mathfrak{E}$ will always be considered as a topological vector space, the topology being given by the zero-neighbourhood basis $\mathfrak{U}_\gamma := \{\mathfrak{y} \in \mathfrak{E} \mid \varphi\langle\mathfrak{y}\rangle \geqslant \gamma\}, \gamma \in \Gamma$. If $(\mathfrak{e}_\iota)_{\iota \in I}$ is any family over vectors in $\mathfrak{E}$ such that the net of all *finite* ("partial") sums $\sum \mathfrak{e}_\iota$ has a limit $\mathfrak{x}$ in $\mathfrak{E}$ then we write $\mathfrak{x} = \sum_{\iota \in I} \mathfrak{e}_\iota$ and call $(\mathfrak{e}_\iota)_{\iota \in I}$ summable.

LEMMA 16. *Let* $(\mathfrak{e}_\iota)_{\iota \in I}$ *be an orthogonal family in the definite space* $(\mathfrak{E}; \langle\ ,\ \rangle)$ *and* $\mathfrak{F}$ *its span. For each* $\mathfrak{x}$ *in the topological closure of* $\mathfrak{F}$ *we have* $\mathfrak{x} = \sum_{\iota \in I} \langle \mathfrak{x}, \mathfrak{e}_\iota\rangle \langle \mathfrak{e}_\iota\rangle^{-1} \mathfrak{e}_\iota$.

Proof. Let $\mathscr{P}$ be the set of all finite subsets of I. For $V \in \mathscr{P}$ we set $\mathfrak{x}_V := \sum_{\iota \in V} \langle \mathfrak{x}, \mathfrak{e}_\iota\rangle \langle \mathfrak{e}_\iota\rangle^{-1} \mathfrak{e}_\iota$. We have to prove that for each $\gamma \in \Gamma$ there is $U \in \mathscr{P}$ such that $\varphi\langle \mathfrak{x} - \mathfrak{x}_V\rangle \geqslant \varepsilon$ for all V *with* $U \subset V \in \mathscr{P}$. Now there is $\mathfrak{y} \in \mathfrak{F}$ with $\varphi(\mathfrak{x} - \mathfrak{y}\rangle \geqslant \varepsilon$. Pick $U \in \mathscr{P}$ with $\mathfrak{y} \in \operatorname{span}\{\mathfrak{e}_\iota \mid \iota \in U\}$. If $U \subset V \in \mathscr{P}$ then $\mathfrak{x} - \mathfrak{x}_V \perp \mathfrak{x}_V - \mathfrak{y}$, so by "Pythagoras" (Lemma 14 (ii)) we obtain $\varepsilon \leqslant \varphi\langle \mathfrak{x} - \mathfrak{y}\rangle = \min\{\varphi\langle \mathfrak{x} - \mathfrak{x}_V\rangle, \varphi\langle \mathfrak{x}_V - \mathfrak{y}\rangle\} \leqslant \varphi\langle \mathfrak{x} - \mathfrak{x}_V\rangle$. □

V. NECESSARY CONDITIONS IN $\mathscr{D}$ FOR $L_c = L_{\perp\perp}$

The principal result of this section is

THEOREM 17 ([20]). *Let* $\mathfrak{E}$ *be an infinite dimensional definite space carrying an admissible topology i.e., the topology mentioned in Definition 15 is admissible in the sense of Definition 1; let furthermore* $(\mathfrak{e}_\iota)_{\iota \in I}$ *be an orthogonal family in* $\mathfrak{E}$ *such that* $(\varphi\langle \mathfrak{e}_\iota\rangle)_{\iota \in I}$ *has a lower bound in* Γ. *Then* $\sum_{\iota \in I} \mathfrak{e}_\iota$ *exists.*

Proof. Let $\mathfrak{F} := \operatorname{span}\{\langle \mathfrak{e}_\iota\rangle^{-1} \mathfrak{e}_\iota - \langle \mathfrak{e}_0\rangle^{-1} \mathfrak{e}_0 \mid \iota \in I\}$. We first wish to show that $\langle \mathfrak{e}_0\rangle^{-1} \mathfrak{e}_0$ is not an element of the topological closure $\overline{\mathfrak{F}}$. Indeed, if γ is a lower bound of $(\varphi\langle e_\iota\rangle)_{\iota\in I}$ and if we let $\mathfrak{x} := \sum_{\iota\in U} \lambda_\iota(\langle e_\iota\rangle^{-1}e_\iota - \langle e_0\rangle^{-1}e_0)$ be a typical vector of $\mathfrak{F}$ (U some finite nonvoid subset of $I\backslash\{0\}$) then we get the inequalities

$$\begin{aligned}\varphi\langle \mathfrak{x}-\langle e_0\rangle^{-1}e_0\rangle &= \varphi\langle(-1-\sum_U\lambda_\iota)\langle e_0\rangle^{-1}e_0 + \sum\lambda_\iota\langle e_\iota\rangle^{-1}e_\iota\rangle \\ &= \min_{\iota\in U}\{2\varphi(-1-\sum_U\lambda_\iota)-\varphi\langle e_0\rangle, 2\varphi(\lambda_\iota)-\varphi\langle e_\iota\rangle\} \\ &\leqslant 2\min_{\iota\in U}\{\varphi(-1-\sum_U\lambda_\iota), \varphi(\lambda_\iota)\} - \gamma \leqslant \varphi(-1)-\gamma = -\gamma.\end{aligned}$$

Thus $\overline{\mathfrak{F}} \neq \mathfrak{E}$.

Since $\mathfrak{F}^{\perp\perp} = \overline{\mathfrak{F}}$ we have $\mathfrak{F}^{\perp} \neq (0)$. Pick a non-zero $\mathfrak{x}\in\mathfrak{F}^{\perp}$; so $\langle \mathfrak{x}, e_\iota\rangle\langle e_\iota\rangle^{-1} = \langle \mathfrak{x}, e_0\rangle\langle e_0\rangle^{-1}$. If we assume that $(e_\iota)_{\iota\in I}$ is a *maximal* orthogonal family then by $L_c = L_s$ and Lemma 16 $\mathfrak{x} = \sum_I \langle \mathfrak{x}, e_\iota\rangle\langle e_\iota\rangle^{-1}e_\iota = \langle \mathfrak{x}, e_0\rangle\langle e_0\rangle^{-1}\sum_I e_\iota$ and thus $\sum_I e_\iota \in \mathfrak{F}^{\perp}$. If $(e_\iota)_{\iota\in I}$ is not maximal then we write it as a difference of two maximal bounded families: Complete the given family to a maximal orthogonal bounded family $(e_\iota)_{\iota\in J}$, $J \supset I$, by Zorn's Lemma. For $\iota\in J$ let $\alpha_\iota := 1\in k$ when $\iota\in I$ and $\alpha_\iota := 2$ when $\iota\in J\backslash I$. The two families $(2e_\iota)_{\iota\in J}$, $(\alpha_\iota e_\iota)_{\iota\in J}$ are bounded maximal families to which the previous result may be applied. We get $\sum_{\iota\in I} e_\iota = \sum_{\iota\in I}(2e_\iota) - \sum_{\iota\in I}\alpha_\iota e_\iota \in \mathfrak{E}$. □

Corollary 18. *If* $\mathfrak{E}$ *and* $(e_\iota)_{\iota\in I}$ *are as in Theorem 17 then* $(e_\iota)_{\iota\in I}$ *converges to* $0\in\mathfrak{E}$. □

Corollary 19. *If* $\mathfrak{E}$ *is as in Theorem 17 then the cofinality type of* Γ *is* ω_0. *In particular, the topology on* $\mathfrak{E}$ *satisfies the first countability axiom.* □

Corollary 20. *If* $\mathfrak{E}$ *is as in Theorem 17 then all orthogonal families of non-zero vectors are countable.*

Proof. Let $(e_\iota)_{\iota\in I}$ be such a family; by multiplying e_ι by a suitable scalar, if necessary, we may assume $(\varphi\langle e_\iota\rangle)_{\iota\in I}$ to be bounded below. Since $\sum_{\iota\in I} e_\iota$ exists by Theorem 17, the sets $I_\gamma = \{\iota\in I \mid \varphi\langle e_\iota\rangle \leqslant \gamma\}$ are finite for all $\gamma\in\Gamma$. Let $(\gamma_i)_{i\in\mathbf{N}}$ be confinal in Γ. Then $I = \cup\{I_{\gamma_i} \mid i\in\mathbf{N}\}$ is countable. □

Definition 21. The elements of the group $\Gamma/2\Gamma$ are called *types*. Let $T:\Gamma\to\Gamma/2\Gamma$ be the canonical projection. $T\circ\varphi$ is constant on the square classes of k (elements of $\dot{k}/\dot{k}^2$) and $T\circ\varphi\circ\langle\ \rangle$ is constant on the "punctured" straight lines in E. A family $(\mathfrak{e}_\iota)_{\iota \in I}$ of vectors in $\mathfrak{E}$ is said to satisfy the *type-condition* iff for all $(\alpha_\iota)_{\iota \in I} \in k^I$ the following holds: if $(\varphi\langle\alpha_\iota \mathfrak{e}_\iota\rangle)_{\iota \in I}$ is bounded (below) then $(\alpha_\iota \mathfrak{e}_\iota)_{\iota \in I}$ converges to $0 \in E$.

COROLLARY 22. *Let* $\mathfrak{E}$ *be as in Theorem 17.* $\Gamma/2\Gamma$ *is infinite. Each orthogonal family in* $\mathfrak{E}$ *satisfies the type-condition, equivalently,* $\Gamma/2\Gamma$ *satisfies (8) below.* □

COROLLARY 23. *Let* $\mathfrak{E}$ *be as in Theorem 17. Then* k *is complete.*

Proof. By Corollary 19 it suffices to show that a sequence $(\alpha_i)_{i \in \mathbf{N}}$ with limit $0 \in k$ is summable. Let $(\mathfrak{e}_i)_{i \in \mathbf{N}}$ be maximal orthogonal in $\mathfrak{E}$ with $(\varphi\langle\mathfrak{e}_i\rangle)_{i \in \mathbf{N}}$ bounded below. If $(\lambda_i)_{i \in \mathbf{N}} \in k^{\mathbf{N}}$ has $(\varphi(\lambda_i))_{i \in \mathbf{N}}$ bounded below then $(\lambda_i \mathfrak{e}_i)_{i \in \mathbf{N}}$ is summable and by continuity of $\langle\ ,\ \rangle$ we obtain

$$\left\langle \sum_{\mathbf{N}} \lambda_i \mathfrak{e}_i, \sum_{\mathbf{N}} \mathfrak{e}_i \right\rangle = \sum_{\mathbf{N}} \lambda_i \langle \mathfrak{e}_i \rangle .$$

Thus, all families $(\lambda_i\langle\mathfrak{e}_i\rangle)_{i \in \mathbf{N}}$ with bounded $(\lambda_i)_{i \in \mathbf{N}}$ are summable.

Pick a strictly monotonic sequence $(n_i)_{i \in \mathbf{N}} \in \mathbf{N}^{\mathbf{N}}$ with $u_0 = 0$ and for all $i \in \mathbf{N}^+$ and all $m \geqslant n_i$: $\varphi(\alpha_m) > \varphi\langle\mathfrak{e}_i\rangle$, and set $A_i := \sum \{\alpha_j \mid n_i \leqslant j < n_{i+1}\}$. The family $(A_i)_{i \in \mathbf{N}}$ is summable if and only if $(\alpha_i)_{i \in \mathbf{N}}$ summable and, if the sums exist, these must be equal. If we set $\lambda_i := A_i\langle\mathfrak{e}_i\rangle^{-1}$ then, by what we have shown, the family of the $A_i = \lambda_i\langle\mathfrak{e}_i\rangle$ is summable. □

COROLLARY 24. *Let* $\mathfrak{E}$ *be as in Theorem 17. Then* $\mathfrak{E}$ *is complete.*

Proof. Let $(\mathfrak{x}_i)_{i \in \mathbf{N}}$ be a Cauchy sequence (Corollary 19). For each fixed $\mathfrak{y} \in \mathfrak{E}$ the map $\mathfrak{x} \mapsto \langle\mathfrak{y}, \mathfrak{x}\rangle$ is uniformly continuous. Hence by Cor. 23 the map $f : \mathfrak{y} \mapsto \lim_i \langle\mathfrak{y}, \mathfrak{x}_i\rangle$ is well-defined. As it is a continuous linear map, its kernel is a closed hyper-plane and so $\left(L_c(\mathfrak{E}) = L_{\perp\perp}(\mathfrak{E})\right)$ there is $\mathfrak{a} \in \mathfrak{E}$ such that $f(\mathfrak{y}) = \langle\mathfrak{x}, \mathfrak{a}\rangle$. Let $N \subseteq \mathbf{N}$ be infinite. Because $\lim \varphi\langle\mathfrak{y}, \mathfrak{a} - \mathfrak{x}_i\rangle = \infty$ for all $\mathfrak{y} \in \mathfrak{E}$ it follows by systematic use of the Cauchy-Schwarz inequality that $\{\varphi\langle\mathfrak{a} - \mathfrak{x}_i\rangle \mid i \in N\}$ is not bounded above by any $\gamma \in \Gamma$. Therefore $(\mathfrak{x}_i)_{i \in \mathbf{N}}$ converges to $\mathfrak{a}$.

VI. SUFFICIENT CONDITIONS IN $\mathscr{D}$ FOR $L_c = L_{\perp\perp}$

VI.1. ASSUMPTIONS. In this chapter $(\mathfrak{E}; \langle\ ,\ \rangle)$ is a definite space in the sense of Definition 15. Of the base field k we shall furthermore assume