VI. Sufficient conditions in D for $L_c = L_{\bot \bot}$

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **31 (1985)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

straight lines in E. A family $(\mathfrak{e}_\iota)_{\iota\in I}$ of vectors in $\mathfrak{E}$ is said to satisfy the *type-condition* iff for all $(\alpha_\iota)_{\iota\in I} \in k^I$ the following holds: if $(\varphi\langle\alpha_\iota\mathfrak{e}_\iota\rangle)_{\iota\in I}$ is bounded (below) then $(\alpha_\iota\mathfrak{e}_\iota)_{\iota\in I}$ converges to $0 \in E$.

COROLLARY 22. *Let $\mathfrak{E}$ be as in Theorem 17. $\Gamma/2\Gamma$ is infinite. Each orthogonal family in $\mathfrak{E}$ satisfies the type-condition, equivalently, $\Gamma/2\Gamma$ satisfies (8) below.* □

COROLLARY 23. *Let $\mathfrak{E}$ be as in Theorem 17. Then k is complete.*

Proof. By Corollary 19 it suffices to show that a sequence $(\alpha_i)_{i\in\mathbf{N}}$ with limit $0 \in k$ is summable. Let $(\mathfrak{e}_i)_{i\in\mathbf{N}}$ be maximal orthogonal in $\mathfrak{E}$ with $(\varphi\langle\mathfrak{e}_i\rangle)_{i\in\mathbf{N}}$ bounded below. If $(\lambda_i)_{i\in\mathbf{N}} \in k^{\mathbf{N}}$ has $(\varphi(\lambda_i))_{i\in\mathbf{N}}$ bounded below then $(\lambda_i\mathfrak{e}_i)_{i\in\mathbf{N}}$ is summable and by continuity of $\langle\ ,\ \rangle$ we obtain

$$\left\langle \sum_{\mathbf{N}} \lambda_i\mathfrak{e}_i, \sum_{\mathbf{N}} \mathfrak{e}_i \right\rangle = \sum_{\mathbf{N}} \lambda_i\langle\mathfrak{e}_i\rangle .$$

Thus, all families $(\lambda_i\langle\mathfrak{e}_i\rangle)_{i\in\mathbf{N}}$ with bounded $(\lambda_i)_{i\in\mathbf{N}}$ are summable.

Pick a strictly monotonic sequence $(n_i)_{i\in\mathbf{N}} \in \mathbf{N}^{\mathbf{N}}$ with $u_0 = 0$ and for all $i \in \mathbf{N}^+$ and all $m \geqslant n_i$: $\varphi(\alpha_m) > \varphi\langle\mathfrak{e}_i\rangle$, and set $A_i := \sum\{\alpha_j \mid n_i \leqslant j < n_{i+1}\}$. The family $(A_i)_{i\in\mathbf{N}}$ is summable if and only if $(\alpha_i)_{i\in\mathbf{N}}$ summable and, if the sums exist, these must be equal. If we set $\lambda_i := A_i\langle\mathfrak{e}_i\rangle^{-1}$ then, by what we have shown, the family of the $A_i = \lambda_i\langle\mathfrak{e}_i\rangle$ is summable. □

COROLLARY 24. *Let $\mathfrak{E}$ be as in Theorem 17. Then $\mathfrak{E}$ is complete.*

Proof. Let $(\mathfrak{x}_i)_{i\in\mathbf{N}}$ be a Cauchy sequence (Corollary 19). For each fixed $\mathfrak{y} \in \mathfrak{E}$ the map $\mathfrak{x} \mapsto \langle\mathfrak{y}, \mathfrak{x}\rangle$ is uniformly continuous. Hence by Cor. 23 the map $f : \mathfrak{y} \mapsto \lim_i \langle\mathfrak{y}, \mathfrak{x}_i\rangle$ is well-defined. As it is a continuous linear map, its kernel is a closed hyper-plane and so $(L_c(\mathfrak{E}) = L_{\perp\perp}(\mathfrak{E}))$ there is $\mathfrak{a} \in \mathfrak{E}$ such that $f(\mathfrak{y}) = \langle\mathfrak{x}, \mathfrak{a}\rangle$. Let $N \subseteq \mathbf{N}$ be infinite. Because $\lim \varphi\langle\mathfrak{y}, \mathfrak{a}-\mathfrak{x}_i\rangle = \infty$ for all $\mathfrak{y} \in \mathfrak{E}$ it follows by systematic use of the Cauchy-Schwarz inequality that $\{\varphi\langle\mathfrak{a}-\mathfrak{x}_i\rangle \mid i \in N\}$ is not bounded above by any $\gamma \in \Gamma$. Therefore $(\mathfrak{x}_i)_{i\in\mathbf{N}}$ converges to $\mathfrak{a}$.

VI. SUFFICIENT CONDITIONS IN $\mathscr{D}$ FOR $L_c = L_{\perp\perp}$

VI.1. ASSUMPTIONS. In this chapter $(\mathfrak{E}; \langle\ ,\ \rangle)$ is a definite space in the sense of Definition 15. Of the base field k we shall furthermore assume

(cf. Corolllaries 22 and 23)

$\Gamma/2\Gamma$ contains a sequence $(\xi_i+2\Gamma)_{i\in\mathbf{N}}$ such that each

(8) system of representatives $(\xi_i+2\gamma_i)_{i\in\mathbf{N}}$ that is bounded below tends to ∞.

(9) k is complete.

Thus, by (8), $\Gamma/2\Gamma$ will be infinite and the topology on k will satisfy the first countability axiom. There are many fields that satisfy (8) and (9): See Remark 30.

The results in the next sections will culminate in Theorem 28 which characterizes certain definite spaces that are orthomodular.

VI.2. Counting types. Let $\mathfrak{E}$ be the completion of an $\aleph_0$-dimensional space $\mathfrak{F}$ which is spanned by an orthogonal basis $\mathscr{B} = (\mathfrak{e}_i)_{i\in\mathbf{N}}$ that satisfies the type condition (Def. 21). $\mathfrak{F}$ is dense in $\mathfrak{E}$ so $\mathfrak{F}^{\perp} = (0)$ and hence $\mathscr{B}$ is maximal. By Lemma 16 we have therefore $\mathfrak{x} = \sum_{\mathbf{N}} \langle \mathfrak{x}, \mathfrak{e}_i\rangle \langle \mathfrak{e}_i\rangle^{-1}\mathfrak{e}_i$ for all $\mathfrak{x}\in\mathfrak{E}$.

We now introduce the function ν which counts types on $\mathscr{B}$. Let $\nu\colon \Gamma/2\Gamma \to \mathbf{N}: t \mapsto \operatorname{card}\{i\in\mathbf{N} \mid T\circ\varphi\langle\mathfrak{e}_i\rangle = t\}$ (cf. Def. 21). We have

Lemma 25. *If* $\mathfrak{f}_1, \ldots, \mathfrak{f}_m$ *are pairwise orthogonal (non zero) vectors in* $\mathfrak{E}$ *with* $T\circ\varphi\langle\mathfrak{f}_i\rangle = t\in\Gamma/2\Gamma$ *for all* $1\leqslant i\leqslant m$ *then* $m\leqslant\nu(t)$.

Proof. We shall replace the $\mathfrak{f}_i$ by suitable multiples and assume that $\varphi\langle\mathfrak{f}_i\rangle = \gamma\in\Gamma$ for all $1\leqslant i\leqslant m$. Let $J := \{i\in\mathbf{N} \mid T\circ\varphi\langle\mathfrak{e}_i\rangle = t\}$. We have $\mathfrak{f}_j = \mathfrak{f}'_j + \mathfrak{f}''_j$ where

$$\mathfrak{f}'_j := \sum_J \langle\mathfrak{f}_j, \mathfrak{e}_i\rangle\langle\mathfrak{e}_i\rangle^{-1}\mathfrak{e}_i, \qquad \mathfrak{f}''_j := \sum_{\mathbf{N}\setminus J} \langle\mathfrak{f}_i, \mathfrak{e}_i\rangle\langle\mathfrak{e}_i\rangle^{-1}\mathfrak{e}_i .$$

Since Lemma 14 (ii) generalizes to finite as well as to infinite sums we find $\varphi\langle\mathfrak{f}''_j\rangle = \min_{i\in\mathbf{N}\setminus J}\{\varphi\langle\langle\mathfrak{f}_j, \mathfrak{e}_i\rangle\langle\mathfrak{e}_i\rangle^{-1}\mathfrak{e}_i\rangle\} \neq \varphi\langle\mathfrak{f}_j\rangle$ (because types are different). By Lemma 14 (ii) furthermore $\varphi\langle\mathfrak{f}_j\rangle\leqslant\varphi\langle\mathfrak{f}'_j\rangle$, $\varphi\langle\mathfrak{f}_j\rangle\leqslant\varphi\langle\mathfrak{f}''_j\rangle$ and we must have equality in at least one instance. Therefore

$$\varphi\langle\mathfrak{f}_j\rangle = \varphi\langle\mathfrak{f}'_j\rangle = \gamma < \varphi\langle\mathfrak{f}''_j\rangle, \qquad 1\leqslant j\leqslant m \tag{10}$$

Now, for $i\neq j$ we find

$$\begin{aligned} 2\varphi\langle\mathfrak{f}'_i, \mathfrak{f}'_j\rangle &= 2\varphi\langle\mathfrak{f}_i-\mathfrak{f}''_i, \mathfrak{f}_j-\mathfrak{f}''_j\rangle \geqslant \min\{2\varphi\langle\mathfrak{f}_i, \mathfrak{f}''_j\rangle, 2\varphi\langle\mathfrak{f}''_i, \mathfrak{f}_j\rangle, 2\varphi\langle\mathfrak{f}''_i, \mathfrak{f}''_j\rangle\} \\ &\geqslant \min\{\varphi\langle\mathfrak{f}_i\rangle + \varphi\langle\mathfrak{f}''_j\rangle, \varphi\langle\mathfrak{f}''_i\rangle + \varphi\langle\mathfrak{f}_j\rangle, \varphi\langle\mathfrak{f}''_i\rangle + \varphi\langle\mathfrak{f}''_j\rangle\} > 2\gamma \end{aligned}$$

so that

$$\varphi\langle \mathfrak{f}'_i, \mathfrak{f}'_j \rangle > \gamma, \quad 1 \leqslant i \neq j \leqslant m \tag{11}$$

Thus $\mathfrak{f}'_1, ..., \mathfrak{f}'_m$ are an almost orthogonal system in the $\nu(t)$-dimensional space $k(\mathfrak{e}_i)_{i \in J}$. Assume by way of contradiction that the $\mathfrak{f}'_j$ were linearly dependent, $\sum_1^m \mu_i \mathfrak{f}'_i = 0$ and not all $\mu_i = 0$. Thus, for each

$$r \in \{1, ..., m\}, 0 = \sum \mu_i \langle \mathfrak{f}'_i, \mathfrak{f}'_r \rangle$$

and so for each r

$$\varphi\langle \mathfrak{f}'_r \rangle + \varphi(\mu_r) = \varphi(-\sum_{j \neq r} \mu_j \langle \mathfrak{f}'_j, \mathfrak{f}'_r \rangle) \geqslant \min_{j \neq r} \{\varphi(\mu_j) + \varphi\langle \mathfrak{f}'_j, \mathfrak{f}'_r \rangle\}.$$

Therefore, by (10) and (11), $\varphi(\mu_r) > \min_{j \neq r} \{\varphi(\mu_j)\}$ which tells that there is no smallest $\varphi(\mu_r)$ at all, a contradiction. Therefore, $\mathfrak{f}'_1, ..., \mathfrak{f}'_m$ are linearly independent and so $m \leqslant \nu(t)$, QED. By Lemma 27 we thus obtain

COROLLARY 26. *The function* ν *that counts types on an orthogonal basis of* $\mathfrak{E}$ *is the same on all bases.*

VI.3. THE TYPE CONDITION. Let $\mathfrak{E}$ be the completion of a $\aleph_0$-dimensional space $\mathfrak{F}$ which is spanned by an orthogonal basis $(\mathfrak{e}_i)_{i \in \mathbf{N}}$ that satisfies the type condition (Def. 21).

LEMMA 27. *Let* $\mathscr{B} = (\mathfrak{u}_i)_{i \in \mathbf{N}}$ *be a maximal orthogonal family in* $\mathfrak{E}$. *Then* $\mathscr{B}$ *satisfies the type condition and* $\mathfrak{x} = \sum_{\mathbf{N}} \langle \mathfrak{x}, \mathfrak{u}_i \rangle \langle \mathfrak{u}_i \rangle^{-1} \mathfrak{u}_i$ *for all* $\mathfrak{x} \in \mathfrak{E}$. *In particular, the span of* $\mathscr{B}$ *is dense in* $\mathfrak{E}$.

Proof. The assertion on the type condition follows directly from Lemma 25. Let then $\mathfrak{x} \in \mathfrak{E}$.

$$\begin{aligned}\varphi\langle \langle \mathfrak{x}, \mathfrak{u}_i \rangle \langle \mathfrak{u}_i \rangle^{-1} \mathfrak{u}_i \rangle = 2\varphi\langle \mathfrak{x}, \mathfrak{u}_i \rangle - \varphi\langle \mathfrak{u}_i \rangle \geqslant \varphi\langle \mathfrak{x} \rangle \\ + \varphi\langle \mathfrak{u}_i \rangle - \varphi\langle \mathfrak{u}_i \rangle = \varphi\langle \mathfrak{x} \rangle.\end{aligned}$$

Thus the family of vectors $\langle \mathfrak{x}, \mathfrak{u}_i \rangle \langle \mathfrak{u}_i \rangle^{-1} \mathfrak{u}_i$ is bounded; in fact, it is a null sequence as $\mathscr{B}$ satisfies the type condition, hence it is summable as $\mathfrak{E}$ is complete. Put $\mathfrak{y} := \sum_{\mathbf{N}} \langle \mathfrak{x}, \mathfrak{u}_i \rangle \langle \mathfrak{u}_i \rangle^{-1} \mathfrak{u}_i$. We have $\langle \mathfrak{u}_i, \mathfrak{y} - \mathfrak{x} \rangle = \langle \mathfrak{u}_i, \mathfrak{y} \rangle - \langle \mathfrak{u}_i, \mathfrak{x} \rangle = 0$, so $\mathfrak{x} - \mathfrak{y} = 0$ as $\mathscr{B}$ is a maximal orthogonal family. □