

# VII. The Main Theorem

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **31 (1985)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **23.05.2024**

## Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

## Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

## VII. THE MAIN THEOREM

We are now able to characterize the definite spaces whose topology is admissible (Def. 1). Refer to Definition 21 for “type condition”.

**THEOREM 28 [20].** *Let  $\mathfrak{E}$  be a definite space in the sense of Definition 15. The following conditions are equivalent*

- (i)  $L_c(\mathfrak{E}) = L_s(\mathfrak{E})$  (cf. (1), (2), (3))
- (ii)  $L_c(\mathfrak{E}) = L_{\perp\perp}(\mathfrak{E})$  (“the topology is admissible”, Def. 1)
- (iii)  $k$  is complete and  $\mathfrak{E}$  is the completion of a  $\aleph_0$ -dimensional space spanned by an orthogonal basis that satisfies the type condition.

*Proof.* (i)  $\Rightarrow$  (ii) holds trivially because  $L_s \subseteq L_{\perp\perp} \subseteq L_c$  by continuity of the form; (ii)  $\Rightarrow$  (iii) was carried out in Chapter V. Just as in [18] we can establish (iii)  $\Rightarrow$  (i). Let  $\mathfrak{U} \in L_c(\mathfrak{E})$ . Pick a maximal orthogonal family  $(v_i)_{i \in I}$  in  $\mathfrak{U}$  and extend it to a maximal orthogonal family  $(v_i)_{I \cup J}$  in  $\mathfrak{E}$ . For  $x \in \mathfrak{E}$  we have by Lemma 27  $x = x' + x''$  where  $x' = \sum_I \langle x, v_i \rangle \langle v_i \rangle^{-1} v_i$  and  $x'' = \sum_J \langle x, v_i \rangle \langle v_i \rangle^{-1} v_i$ . Now  $x' \in \overline{\mathfrak{U}} = \mathfrak{U}$ . All that remains to be shown is  $x'' \in \mathfrak{U}^\perp$ . Now  $\mathfrak{U}^\perp$  is closed so it suffices to show that  $v_i \in \mathfrak{U}^\perp$  for all  $i \in J$ . To this end pick  $u \in \mathfrak{U}$  and decompose  $u = u' + u''$  (analogous to the decomposition of  $x$ ):  $u'' = u - u' \in \mathfrak{U} - \mathfrak{U} = \mathfrak{U}$ . Now  $\langle u'', v_i \rangle = 0$  for all  $i \in I$  so  $u'' = 0$  since  $(v_i)_{i \in I}$  is a maximal orthogonal family. From

$$0 = u'' : = \sum_J \langle u, v_i \rangle \langle v_i \rangle^{-1} v_i$$

we obtain  $\langle u, v_i \rangle = 0$  ( $i \in J$ ). As  $u \in \mathfrak{U}$  was arbitrary this says that  $v_i \in \mathfrak{U}^\perp$  ( $i \in J$ ).

Q.E.D.

*Remark 29.* Let the definite space  $\mathfrak{E}$  be the completion of  $\mathfrak{F} = k(e_i)_{i \in \mathbb{N}}, (e_i)_{\mathbb{N}}$  an orthogonal family (that does not necessarily satisfy the type condition). If  $k$  is complete then  $\mathfrak{E}$  is isometric to the  $k$ -space  $\hat{\mathfrak{F}}$  of all sequences  $(\lambda_i)_{i \in \mathbb{N}} \in k^{\mathbb{N}}$  such that  $\lim_{\mathbb{N}}(2\varphi\lambda_i + \varphi\langle e_i \rangle) = \infty$  and equipped with the form  $\langle (\lambda_i), (\mu_i) \rangle = \sum_{\mathbb{N}} \lambda_i \mu_i \langle e_i \rangle$ . Indeed, the set  $\hat{\mathfrak{F}}$  is a definite  $k$ -space and the map  $\Psi: (\lambda_i) \rightarrow \sum \lambda_i e_i$  is a well defined isometry  $\hat{\mathfrak{F}} \rightarrow \Psi(\hat{\mathfrak{F}}) \subset \mathfrak{E}$ . By the “infinite Pythagoras” we have  $\ker \Psi = 0$ ; on the other hand, Lemma 16 shows that  $\Psi$  is also surjective.

Thus *all* definite spaces that carry an admissible topology are (by Theorem 28) of the kind invented by Keller.

*Remark 30.* By Theorem 28 the isometry type of a definite space with admissible topology is characterized by the sequence  $(\langle e_i \rangle)_{i \in \mathbb{N}}$  where  $(e_i)_{i \in \mathbb{N}}$  is a maximal orthogonal family in  $\mathfrak{E}$ . Conversely, for each  $(\alpha_i) \in k^{\mathbb{N}}$  there is a definite space  $\mathfrak{E}$  with  $L_c(\mathfrak{E}) = L_s(\mathfrak{E})$  admitting a maximal orthogonal family  $(e_i)_{i \in \mathbb{N}}$  with  $\langle e_i \rangle = \alpha_i$  ( $i \in \mathbb{N}$ ) provided that

- (A)  $\xi_i := \varphi \alpha_i \in \Gamma$  satisfies the (type-) condition expressed in (8)
- (B) The form  $\langle \cdot, \cdot \rangle$  defined on  $\mathfrak{F} := k(e_i)_{i \in \mathbb{N}}$  by  $\langle e_i, e_j \rangle = 0$  ( $i \neq j$ ),  $\langle e_i \rangle = \alpha_i$  ( $i \in \mathbb{N}$ ) is definite.

These two conditions are implemented by many fields. In order to satisfy (A) one may, e.g. pick fields of generalized formal power series that are complete under a valuation  $\varphi$  with group  $\Gamma$  a prescribed Hahn product [30, p. 31] with sufficiently many factors not 2-divisible, e.g.  $\Gamma = \mathbf{Z}^{(\mathbb{N})}$  ordered antilexicographically. Let  $k$  be any field with (A) and  $t \in \Gamma/2\Gamma$ ; set  $\mathfrak{F}_t = \{\text{span } e_i \mid \varphi \alpha_i + 2\Gamma = t\}$ . By (A)  $\dim \mathfrak{F}_t < \infty$ ; furthermore

$$\mathfrak{F} = \bigoplus^{\perp} \{\mathfrak{F}_t \mid t \in \Gamma/2\Gamma\}.$$

In order to check whether the form  $\langle \cdot, \cdot \rangle$  satisfies the triangle inequality on  $\mathfrak{F}$  it suffices to verify said inequality on each  $\mathfrak{F}_t$ . A. Fässler has given a handy criterium for  $\langle \cdot, \cdot \rangle$  to be definite if Hahnproducts  $\Gamma$  are used, as indicated, to construct  $k$  with (A), [6, Lemma 15, 16].

### VIII. APPENDIX: EXTENDING THE MAIN THEOREM TO THE CLASS $\mathcal{E}$ OF NORM-TOPOLOGICAL SPACES

The arguments applied to the spaces in the class  $\mathcal{D}$  can be extended to a larger class  $\mathcal{E}$ . First we have (cf. Definition 15):

*Definition 31.* An infinite dimensional anisotropic quadratic space  $(\mathfrak{E}; \langle \cdot, \cdot \rangle)$  over a \*-valued field  $(k, *, \varphi, \Gamma)$  is called norm-topological if the sets  $\mathfrak{U}_{\gamma} := \{\mathfrak{x} \in \mathfrak{E} \mid \varphi \langle \mathfrak{x} \rangle > \gamma\}$  form a 0-neighbourhood basis of a vector space topology on  $\mathfrak{E}$ . Let  $\mathcal{E}$  be the class of all norm-topological spaces.

Definite spaces are norm-topological, obviously.

A proper subgroup  $\Delta$  of  $\Gamma$  is *convex* (or isolated) if “ $0 \leqslant x \leqslant y \& y \in \Delta$ ” implies “ $x \in \Delta$ ”. If the subgroup  $\Delta \subset \Gamma$  is convex then the factor group  $\Gamma/\Delta$  is ordered by setting  $\gamma + \Delta \leqslant \delta + \Delta$  iff  $\gamma < \delta$  or  $\gamma - \delta \in \Delta$ ; furthermore,  $\varphi_{\Delta}: k \rightarrow \Gamma/\Delta \cup \{\infty\}$  defined by  $\varphi_{\Delta}(\alpha) = \varphi(\alpha) + \Delta$  is a valuation (a “coarser valuation”) which yields the same topology on  $k$  as  $\varphi$ .