# X. Clifford algebras of orthomodular spaces

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## X. Clifford algebras of orthomodular spaces

X.1. Assumptions. In Chap. X $k$ is a commutative field of characteristic not 2 and $\langle\ ,\ \rangle$ is a symmetric bilinear anisotropic form $\mathfrak{E} \times \mathfrak{E} \to k$ on the $k$-vector space $\mathfrak{E}$.

$C(\mathfrak{E})$ is the Clifford algebra of $(\mathfrak{E}; \langle\ ,\ \rangle)$; it is a $k$-algebra that contains the space $\mathfrak{E}$ as a set of ring generators which satisfy $\mathfrak{x} \cdot \mathfrak{y} + \mathfrak{y} \cdot \mathfrak{x} = 2\langle \mathfrak{x}, \mathfrak{y}\rangle$. For any pair of elements $\mathfrak{c}, \mathfrak{d} \in C(\mathfrak{E})$ there exists a finite orthogonal family $\mathfrak{e}_0, \ldots, \mathfrak{e}_n$ in $\mathfrak{E}$ such that $\mathfrak{c} = \sum_I \alpha_I \mathfrak{e}_I$, $\mathfrak{d} = \sum \beta_I \mathfrak{e}_I$; here the summation index $I$ runs over all subsets

$I = \{\iota_1 < \ldots < \iota_r\}$ of $\{0, 1, \ldots, n\}$ and $\mathfrak{e}_I := \mathfrak{e}_{\iota_1} \cdot \mathfrak{e}_{\iota_2} \cdot \ldots \cdot \mathfrak{e}_{\iota_r}$; the empty product $\mathfrak{e}_\varnothing$ is the unit element in $C(\mathfrak{E})$.

There is a *canonical* symmetric bilinear form $\langle\ ,\ \rangle$ on $C(\mathfrak{E})$ which extends the given form on $\mathfrak{E}$ ([5, 11, 22]). One has

$$\langle \mathfrak{c}, \mathfrak{d}\rangle = \sum_I \alpha_I \beta_I \prod_{\iota \in I} \langle \mathfrak{e}_\iota, \mathfrak{e}_\iota\rangle \tag{16}$$

From now on we shall assume that $(\mathfrak{E}; \langle\ ,\ \rangle)$ is an infinite dimensional definite space.

X.2. Clifford algebras of definite spaces. In [6] Angela Fässler has proved that for certain definite orthomodular spaces $\mathfrak{E}$ the algebra $C(\mathfrak{E})$ is a skew field; furthermore, the $k$-vector space $C(\mathfrak{E})$ equipped with the form (16) is a definite space whose completion $\tilde{C}(\mathfrak{E})$ is orthomodular again. Furthermore $\tilde{C}(\mathfrak{E})$ is a skew field, in fact, a *-valued field with * the extension to $\tilde{C}(\mathfrak{E})$ of the main antiautomorphism of the Clifford algebra $C(\mathfrak{E})$; the residue class field of $\tilde{C}(\mathfrak{E})$ is isomorphic to the residue class field of $\varphi$.

In the following theorem we prove the main fact in a simplified and slightly more general setting.

Theorem 37. *Assume that in the definite space* $(\mathfrak{E}; \langle\ ,\ \rangle)$ *each orthogonal family* $\mathfrak{e}_0, \ldots \mathfrak{e}_n$ *has*

$$\varphi\langle \mathfrak{e}_0\rangle + \ldots + \varphi\langle \mathfrak{e}_n\rangle \notin 2\Gamma \tag{17}$$

*Then:*

(i) $C(\mathfrak{E})$ *equipped with the form in (16) is a definite space,*

(ii) $C(\mathfrak{E})$ *is a division ring,*

(iii) *The map* $\tilde{\varphi}: C(\mathfrak{E}) \to \Gamma \cup \{\infty\}$ *defined by* $\mathfrak{c} \mapsto \varphi\langle\mathfrak{c}\rangle$ *is a* *-*valuation for* $*$ *the main antiautomorphism of* $C(\mathfrak{E})$.

*Proof.* (i) It suffices to prove the triangle inequality (Lemma 14 (i)). Write $\mathfrak{c} = \sum \alpha_I \mathfrak{e}_I$, $\mathfrak{d} = \sum \beta_I \mathfrak{e}_I$ as in X.1. Then we have $\varphi\langle\alpha\mathfrak{e}_I\rangle \neq \varphi\langle\beta\mathfrak{e}_J\rangle$ for $I \neq J$ and $\alpha \neq 0 \neq \beta$. Hence

$$\varphi\langle\mathfrak{c}\rangle = \varphi\sum\langle\alpha_I\mathfrak{e}_I\rangle = \min_I \{\varphi\langle\alpha_I\mathfrak{e}_I\rangle\}$$

and similarly for $\varphi\langle\mathfrak{d}\rangle$. Therefore

$$\varphi\langle\mathfrak{c}+\mathfrak{d}\rangle = \varphi\sum\langle(\alpha_I+\beta_I)\mathfrak{e}_I\rangle \geqslant \min\{2\varphi(\alpha_I+\beta_I) + \varphi\langle\mathfrak{e}_I\rangle\}$$
$$\geqslant \min\{2\varphi\alpha_I + \varphi\langle\mathfrak{e}_I\rangle, 2\varphi\beta_I + \varphi\langle\mathfrak{e}_I\rangle\} = \min\{\varphi\langle\mathfrak{c}\rangle, \varphi\langle\mathfrak{d}\rangle\}.$$

This proves (i). Next we show

$$\varphi\langle\mathfrak{c}\cdot\mathfrak{d}\rangle = \varphi\langle\mathfrak{c}\rangle + \varphi\langle\mathfrak{d}\rangle \tag{18}$$

Indeed, from

$$\langle\mathfrak{e}_I\cdot\mathfrak{e}_J\rangle = \langle\pm\langle\mathfrak{e}_{I\cap J}\rangle\mathfrak{e}_{(I\cup J)\setminus(I\cap J)}\rangle = \langle\mathfrak{e}_{I\cap J}\rangle^2\langle\mathfrak{e}_{(I\cup J)\setminus(I\cap J)}\rangle = \langle\mathfrak{e}_I\rangle\cdot\langle\mathfrak{e}_J\rangle$$

we see that

$$\varphi\langle\alpha_{I'}\,\mathfrak{e}_{I'}\rangle \leqslant \varphi\langle\alpha_I\mathfrak{e}_I\rangle \;\&\; \varphi\langle\beta_{J'}\,\mathfrak{e}_{J'}\rangle \leqslant \varphi\langle\beta_J\mathfrak{e}_J\rangle$$

implies

$$\varphi\langle\alpha_I, \beta_J, \mathfrak{e}_I, \mathfrak{e}_{J'}\rangle \leqslant \varphi\langle\alpha_I\beta_J\mathfrak{e}_I\mathfrak{e}_J\rangle.$$

We therefore pick $G, H \subseteq \{0, \ldots, n\}$ such that for all $I \subset \{0, \ldots, n\}$ we shall have

$$\varphi\langle\alpha_G\mathfrak{e}_G\rangle \leqslant \varphi\langle\alpha_I\mathfrak{e}_I\rangle, \varphi\langle\beta_H\mathfrak{e}_H\rangle \leqslant \varphi\langle\beta_I\mathfrak{e}_I\rangle.$$

It now follows that

$$\varphi\langle\mathfrak{c}\cdot\mathfrak{d}\rangle = \varphi\langle(\textstyle\sum\alpha_I\mathfrak{e}_I)\cdot\sum\beta_J\mathfrak{e}_J\rangle = \varphi\langle\sum\alpha_I\beta_J\mathfrak{e}_I\mathfrak{e}_J\rangle = \varphi\langle\alpha_G\beta_H\mathfrak{e}_G\mathfrak{e}_H$$
$$+ \textstyle\sum'\alpha_I\beta_J\mathfrak{e}_I\mathfrak{e}_J\rangle = \varphi\langle\alpha_G\beta_H\mathfrak{e}_G\mathfrak{e}_H\rangle = \varphi\langle\mathfrak{c}\rangle + \varphi\langle\mathfrak{d}\rangle.$$

Thus (18) is established.

From (18) it follows that $C(\mathfrak{E})$ has no zero divisors, hence $C(\mathfrak{E})$ is a division ring (being an inductive limit of finite dimensional algebras). The map $\tilde{\varphi}: C(\mathfrak{E}) \to \Gamma \cup \{\infty\}$ as defined in (iii) of the Theorem is a *-valuation, for $\tilde{\varphi}(\mathfrak{c}^*) = \tilde{\varphi}(\mathfrak{c})$ is obvious and everything else has been established already.

COROLLARY 38. *Assume that the definite space* $(\mathfrak{E}; \langle\ ,\ \rangle)$ *is complete and that the system of types (Corollary 26) is linearly independent in* $\Gamma/2\Gamma$ *(considered as a* $\mathbf{Z}_2$*-vector space) then the conclusions (i), (ii), (iii) of Theorem 37 hold.*

$C(\mathfrak{E})$ in Theorem 37 is not complete (unless finite dimensional). Its quadratic form $\langle\ ,\ \rangle$ can be extended to the completion $\tilde{C}$. By using Theorem 28 one can see that this completion has $L_{\perp\perp}(\tilde{C}) = L_c(\tilde{C})$ if and only if $E$ has $L_{\perp\perp}(E) = L_c(E)$.

## XI. CONTINUOUS OPERATORS ARE NOT ALWAYS BOUNDED

XI.1. INTRODUCTION. Let $\mathfrak{E}$ be an infinite dimensional definite space in the sense of Definition 15. A linear map (operator) $h: \mathfrak{E} \to \mathfrak{E}$ is called *bounded* iff there exists $\gamma \in \Gamma$ such that for all $\mathfrak{x} \in \mathfrak{E}$ we have $\varphi\langle h\mathfrak{x}\rangle \geqslant \gamma + \varphi\langle \mathfrak{x}\rangle$.

In [6] A. Fässler gave an explicit example of a continuous operator $h$ on an orthomodular space $\mathfrak{E}$ that is not bounded; she also proved a criterion for boundness which is very useful in the study of the algebra $\mathscr{B}(\mathfrak{E})$ of bounded operators $h: \mathfrak{E} \to \mathfrak{E}$ when $\mathfrak{E}$ is an orthomodular definite space of a certain kind. We shall prove this criterion anew here as its original proof can be shortened considerably.

We shall consider definite spaces that satisfy

(19) $(\mathfrak{E}; \langle\ ,\ \rangle)$ contains a maximal orthogonal family $(\mathfrak{e}_i)_{\mathbf{N}}$ such that the groups $\Theta(\varphi\langle \mathfrak{e}_i\rangle)$ are different.

By (14) we see that (19) is a property of $\mathfrak{E}$, not of $(\mathfrak{e}_i)_{\mathbf{N}}$; Keller's original example of an orthomodular space satisfies (19).

XI.2. FÄSSLER'S CRITERION. In this subsection let $(\mathfrak{E}; \langle\ ,\ \rangle)$ be an infinite dimensional orthomodular space that has (19). Fix a maximal orthogonal family $(\mathfrak{e}_i)_{\mathbf{N}}$ that enjoys (19). If $f: \mathfrak{E} \to \mathfrak{E}$ is given, expand (Lemma 27)

(20) $$f\mathfrak{e}_i = \sum_{j\in\mathbf{N}} \alpha_{ij}\mathfrak{e}_j \qquad (i \in \mathbf{N})$$

THEOREM 39 ([6]). *The linear map* $f$ *is bounded iff it is continuous and satisfies*