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## X. CLIFFORD ALGEBRAS OF ORTHOMODULAR SPACES

X.1. ASSUMPTIONS. In Chap. X  $k$  is a commutative field of characteristic not 2 and  $\langle \cdot, \cdot \rangle$  is a symmetric bilinear anisotropic form  $\mathfrak{E} \times \mathfrak{E} \rightarrow k$  on the  $k$ -vector space  $\mathfrak{E}$ .

$C(\mathfrak{E})$  is the Clifford algebra of  $(\mathfrak{E}; \langle \cdot, \cdot \rangle)$ ; it is a  $k$ -algebra that contains the space  $\mathfrak{E}$  as a set of ring generators which satisfy  $\mathfrak{x} \cdot \mathfrak{y} + \mathfrak{y} \cdot \mathfrak{x} = 2\langle \mathfrak{x}, \mathfrak{y} \rangle$ . For any pair of elements  $\mathfrak{c}, \mathfrak{d} \in C(\mathfrak{E})$  there exists a finite orthogonal family  $\mathfrak{e}_0, \dots, \mathfrak{e}_n$  in  $\mathfrak{E}$  such that  $\mathfrak{c} = \sum_I \alpha_I \mathfrak{e}_I, \mathfrak{d} = \sum_I \beta_I \mathfrak{e}_I$ ; here the summation index  $I$  runs over all subsets

$I = \{\iota_1 < \dots < \iota_r\}$  of  $\{0, 1, \dots, n\}$  and  $\mathfrak{e}_I := \mathfrak{e}_{\iota_1} \cdot \mathfrak{e}_{\iota_2} \cdot \dots \cdot \mathfrak{e}_{\iota_r}$ ; the empty product  $\mathfrak{e}_\emptyset$  is the unit element in  $C(\mathfrak{E})$ .

There is a *canonical* symmetric bilinear form  $\langle \cdot, \cdot \rangle$  on  $C(\mathfrak{E})$  which extends the given form on  $\mathfrak{E}$  ([5, 11, 22]). One has

$$(16) \quad \langle \mathfrak{c}, \mathfrak{d} \rangle = \sum_I \alpha_I \beta_I \prod_{\iota \in I} \langle \mathfrak{e}_\iota, \mathfrak{e}_\iota \rangle$$

From now on we shall assume that  $(\mathfrak{E}; \langle \cdot, \cdot \rangle)$  is an infinite dimensional definite space.

X.2. CLIFFORD ALGEBRAS OF DEFINITE SPACES. In [6] Angela Fässler has proved that for certain definite orthomodular spaces  $\mathfrak{E}$  the algebra  $C(\mathfrak{E})$  is a skew field; furthermore, the  $k$ -vector space  $C(\mathfrak{E})$  equipped with the form (16) is a definite space whose completion  $\tilde{C}(\mathfrak{E})$  is orthomodular again. Furthermore  $\tilde{C}(\mathfrak{E})$  is a skew field, in fact, a \*-valued field with \* the extension to  $\tilde{C}(\mathfrak{E})$  of the main antiautomorphism of the Clifford algebra  $C(\mathfrak{E})$ ; the residue class field of  $\tilde{C}(\mathfrak{E})$  is isomorphic to the residue class field of  $\varphi$ .

In the following theorem we prove the main fact in a simplified and slightly more general setting.

THEOREM 37. Assume that in the definite space  $(\mathfrak{E}; \langle \cdot, \cdot \rangle)$  each orthogonal family  $\mathfrak{e}_0, \dots, \mathfrak{e}_n$  has

$$(17) \quad \varphi \langle \mathfrak{e}_0 \rangle + \dots + \varphi \langle \mathfrak{e}_n \rangle \notin 2\Gamma$$

Then:

- (i)  $C(\mathfrak{E})$  equipped with the form in (16) is a definite space,
- (ii)  $C(\mathfrak{E})$  is a division ring,

(iii) *The map  $\tilde{\phi}: C(\mathfrak{E}) \rightarrow \Gamma \cup \{\infty\}$  defined by  $c \mapsto \phi(c)$  is a \*-valuation for \* the main antiautomorphism of  $C(\mathfrak{E})$ .*

*Proof.* (i) It suffices to prove the triangle inequality (Lemma 14 (i)). Write  $c = \sum \alpha_I e_I$ ,  $d = \sum \beta_I e_I$  as in X.1. Then we have  $\phi(\alpha e_I) \neq \phi(\beta e_J)$  for  $I \neq J$  and  $\alpha \neq 0 \neq \beta$ . Hence

$$\phi(c) = \phi \sum_I \langle \alpha_I e_I \rangle = \min_I \{ \phi(\alpha_I e_I) \}$$

and similarly for  $\phi(d)$ . Therefore

$$\begin{aligned} \phi(c+d) &= \phi \sum_I \langle (\alpha_I + \beta_I) e_I \rangle \geq \min \{ 2\phi(\alpha_I + \beta_I) + \phi(e_I) \} \\ &\geq \min \{ 2\phi(\alpha_I) + \phi(e_I), 2\phi(\beta_I) + \phi(e_I) \} = \min \{ \phi(c), \phi(d) \}. \end{aligned}$$

This proves (i). Next we show

$$(18) \quad \phi(c \cdot d) = \phi(c) + \phi(d)$$

Indeed, from

$$\langle e_I \cdot e_J \rangle = \langle \pm \langle e_{I \cap J} \rangle e_{(I \cup J) \setminus (I \cap J)} \rangle = \langle e_{I \cap J} \rangle^2 \langle e_{(I \cup J) \setminus (I \cap J)} \rangle = \langle e_I \rangle \cdot \langle e_J \rangle$$

we see that

$$\phi(\alpha_I e_I) \leq \phi(\alpha_I e_I) \& \phi(\beta_J e_J) \leq \phi(\beta_J e_J)$$

implies

$$\phi(\alpha_I \beta_J e_I e_J) \leq \phi(\alpha_I \beta_J e_I e_J).$$

We therefore pick  $G, H \subseteq \{0, \dots, n\}$  such that for all  $I \subset \{0, \dots, n\}$  we shall have

$$\phi(\alpha_G e_G) \leq \phi(\alpha_I e_I), \phi(\beta_H e_H) \leq \phi(\beta_I e_I).$$

It now follows that

$$\begin{aligned} \phi(c \cdot d) &= \phi((\sum \alpha_I e_I) \cdot (\sum \beta_J e_J)) = \phi(\sum \alpha_I \beta_J e_I e_J) = \phi(\alpha_G \beta_H e_G e_H) \\ &\quad + \sum' \alpha_I \beta_J e_I e_J = \phi(\alpha_G \beta_H e_G e_H) = \phi(c) + \phi(d). \end{aligned}$$

Thus (18) is established.

From (18) it follows that  $C(\mathfrak{E})$  has no zero divisors, hence  $C(\mathfrak{E})$  is a division ring (being an inductive limit of finite dimensional algebras). The map  $\tilde{\phi}: C(\mathfrak{E}) \rightarrow \Gamma \cup \{\infty\}$  as defined in (iii) of the Theorem is a \*-valuation, for  $\tilde{\phi}(c^*) = \tilde{\phi}(c)$  is obvious and everything else has been established already.

COROLLARY 38. Assume that the definite space  $(\mathfrak{E}; \langle \cdot, \cdot \rangle)$  is complete and that the system of types (Corollary 26) is linearly independent in  $\Gamma/2\Gamma$  (considered as a  $\mathbf{Z}_2$ -vector space) then the conclusions (i), (ii), (iii) of Theorem 37 hold.

$C(\mathfrak{E})$  in Theorem 37 is not complete (unless finite dimensional). Its quadratic form  $\langle \cdot, \cdot \rangle$  can be extended to the completion  $\tilde{C}$ . By using Theorem 28 one can see that this completion has  $L_{\perp\perp}(\tilde{C}) = L_c(\tilde{C})$  if and only if  $E$  has  $L_{\perp\perp}(E) = L_c(E)$ .

## XI. CONTINUOUS OPERATORS ARE NOT ALWAYS BOUNDED

XI.1. INTRODUCTION. Let  $\mathfrak{E}$  be an infinite dimensional definite space in the sense of Definition 15. A linear map (operator)  $h: \mathfrak{E} \rightarrow \mathfrak{E}$  is called *bounded* iff there exists  $\gamma \in \Gamma$  such that for all  $x \in \mathfrak{E}$  we have  $\varphi\langle hx \rangle \geq \gamma + \varphi\langle x \rangle$ .

In [6] A. Fässler gave an explicit example of a continuous operator  $h$  on an orthomodular space  $\mathfrak{E}$  that is not bounded; she also proved a criterion for boundness which is very useful in the study of the algebra  $\mathcal{B}(\mathfrak{E})$  of bounded operators  $h: \mathfrak{E} \rightarrow \mathfrak{E}$  when  $\mathfrak{E}$  is an orthomodular definite space of a certain kind. We shall prove this criterion anew here as its original proof can be shortened considerably.

We shall consider definite spaces that satisfy

- (19)  $(\mathfrak{E}; \langle \cdot, \cdot \rangle)$  contains a maximal orthogonal family  $(e_i)_{\mathbb{N}}$  such that the groups  $\Theta(\varphi\langle e_i \rangle)$  are different.

By (14) we see that (19) is a property of  $\mathfrak{E}$ , not of  $(e_i)_{\mathbb{N}}$ ; Keller's original example of an orthomodular space satisfies (19).

XI.2. FÄSSLER'S CRITERION. In this subsection let  $(\mathfrak{E}; \langle \cdot, \cdot \rangle)$  be an infinite dimensional orthomodular space that has (19). Fix a maximal orthogonal family  $(e_i)_{\mathbb{N}}$  that enjoys (19). If  $f: \mathfrak{E} \rightarrow \mathfrak{E}$  is given, expand (Lemma 27)

$$(20) \quad f e_i = \sum_{j \in \mathbb{N}} \alpha_{ij} e_j \quad (i \in \mathbb{N})$$

THEOREM 39 ([6]). *The linear map  $f$  is bounded iff it is continuous and satisfies*