# 1. Endomorphism algebras

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## 1. Endomorphism algebras

In this section $V$ will be an arbitrary module over a commutative ring $R$ with unit, and for each $p \geqslant 0$ $\wedge^p V$ will be its $p^{\text{th}}$ exterior power and $\text{End} \wedge^p V$ will be the $R$-module of endomorphisms $\wedge^p V \to \wedge^p V$; $\Pi_p \text{End} \wedge^p V$ will be the direct product of the $R$-modules $\text{End} \wedge^p V$. We shall define three distinct products in $\Pi_p \text{End} \wedge^p V$; the first two products are standard, and they will be used to define the third product. If $\wedge^p V$ itself vanishes for sufficiently large $p \geqslant 0$ the direct product $\Pi_p \text{End} \wedge^p V$ and the direct sum $\amalg_p \text{End} \wedge^p V$ agree; although this special condition will be satisfied in later sections the definitions in this section will be formulated in complete generality for the direct product $\Pi_p \text{End} \wedge^p V$.

Elements of $\Pi_p \text{End} \wedge^p V$ will be indicated by boldface capital letters $\mathbf{A}, \mathbf{B}, \ldots$, the $p^{\text{th}}$ components being $A_p, B_p, \ldots \in \text{End} \wedge^p V$ for each $p \geqslant 0$. The simplest product in $\Pi_p \text{End} \wedge^p V$ is induced by compositions: the $p^{\text{th}}$ component of the *composition product* $\mathbf{AB} \in \Pi_p \text{End} \wedge^p V$ is the usual composition $A_p B_p \in \text{End} \wedge^p V$ of the endomorphisms $A_p$ and $B_p$ of $\wedge^p V$, where $A_p B_q = 0$ for $p \neq q$. Trivially $\Pi_p \text{End} \wedge^p V$ is an associative $R$-algebra with respect to the composition product, and there is a two-sided unit element $\mathbf{I}$ whose $p^{\text{th}}$ component is the identity endomorphism $I_p \in \text{End} \wedge^p V$ for each $p \geqslant 0$.

There is another reasonably familiar product in $\Pi_p \text{End} \wedge^p V$, the product of $A_p \in \text{End} \wedge^p V$ and $B_q \in \text{End} \wedge^q V$ providing an element

$$A_p \cdot B_q \in \text{End} \wedge^{p+q} V$$

for each $p \geqslant 0$ and each $q \geqslant 0$. Since elements of $\text{End} \wedge^{p+q} V$ are uniquely defined in terms of the behavior on exterior products $x_1 \wedge \ldots \wedge x_{p+q} \in \wedge^{p+q} V$, it suffices to require that

$$(A_p \cdot B_q)(x_1 \wedge \ldots \wedge x_{p+q}) = \sum_\pi \varepsilon_\pi A_p(x_{\pi 1} \wedge \ldots \wedge x_{\pi p}) \wedge B_q(x_{\pi(p+1)} \wedge \ldots \wedge x_{\pi(p+q)})$$

where the sum is computed over all permutations $\pi$ of $\{1, \ldots, p+q\}$ such that both $\pi 1 < \ldots < \pi p$ and $\pi(p+1) < \ldots < \pi(p+q)$, and where $\varepsilon_\pi$ is the parity $\pm 1$ of the permutation $\pi$. Such "shuffle products" $A_p \cdot B_q \in \text{End} \wedge^{p+q} V$ provide a unique *shuffle product* $\mathbf{A} \cdot \mathbf{B} \in \Pi_r \text{End} \wedge^r V$ of any two elements $\mathbf{A}$ and $\mathbf{B}$ in $\Pi_r \text{End} \wedge^r V$.

One easily verifies that the shuffle product is associative and strictly commutative; specifically, $A_p \cdot B_q = B_q \cdot A_p \in \text{End} \wedge^{p+q} V$ with no plus-orminus signs. For example, for $p = 1$ and $q = 1$ one has

$$\begin{aligned}(A_1 \cdot B_1)(x_1 \wedge x_2) &= A_1x_1 \wedge B_1x_2 - A_1x_2 \wedge B_1x_1 \\ &= -B_1x_2 \wedge A_1x_1 + B_1x_1 \wedge A_1x_2 = (B_1 \cdot A_1)(-x_2 \wedge x_1) \\ &= (B_1 \cdot A_1)(x_1 \wedge x_2),\end{aligned}$$

hence $A_1 \cdot B_1 = B_1 \cdot A_1 \in \operatorname{End} \wedge^2 V$. The algebra $\Pi_p \operatorname{End} \wedge^p V$ has a unique (two-sided) unit element with respect to the shuffle product, whose only nonzero component is the identity endomorphism $I_0$ of $\wedge^0 V$.

For any endomorphism $A$ of $V$ itself and any $p \geqslant 0$ there is a well-defined element $A_p \in \operatorname{End} \wedge^p V$ such that

$$A_p(x_1 \wedge \dots \wedge x_p) = Ax_1 \wedge \dots \wedge Ax_p$$

for any $x_1 \wedge \dots \wedge x_p \in \wedge^p V$; in particular $A_1 = A$. Observe that the $p$-fold shuffle product $A^{\cdot p} = A \cdot \dots \cdot A$ is defined by

$$A^{\cdot p}(x_1 \wedge \dots \wedge x_p) = \textstyle\sum_\pi \varepsilon_\pi Ax_{\pi 1} \wedge \dots \wedge Ax_{\pi p},$$

the summation extending overall $p!$ permutations $\pi$ of $\{1, \dots, p\}$. Since $\varepsilon_\pi Ax_{\pi 1} \wedge \dots \wedge Ax_{\pi p} = Ax_1 \wedge \dots \wedge Ax_p$ for each permutation $\pi$ it follows that $A^{\cdot p} = p!\, A_p$. For this reason $A_p$ can reasonably be written $\dfrac{1}{p!} A^{\cdot p}$, without requiring the ground ring to contain the element $\dfrac{1}{p!}$. Thus the direct product of the elements $A_p \left(= \dfrac{1}{p!} A^{\cdot p}\right)$ over all $p \geqslant 0$ is essentially an exponential $e^{\cdot A} \in \Pi_p \operatorname{End} \wedge^p V$. One easily verifies that $e^{\cdot A} \cdot e^{\cdot(-A)} = I_0 = e^{\cdot(-A)} \cdot e^{\cdot A}$, where $I_0 \in \operatorname{End} \wedge^0 V$ represents the unit element in $\Pi_p \operatorname{End} \wedge^p V$ with respect to the shuffle product.

For each $p \geqslant 0$ the $p$-fold shuffle product $I^{\cdot p}$ of the identity endomorphism $I \in \operatorname{End} V$ satisfies $\dfrac{1}{p!} I^{\cdot p} = I_p$, where $I_p$ is the identity endomorphism in $\operatorname{End} \wedge^p V$. Hence $e^{\cdot I}$ is precisely the two-sided unit element $\mathbf{I}$ of $\Pi_p \operatorname{End} \wedge^p V$ with respect to the composition product. Since

$$e^{\cdot I} \cdot e^{\cdot(-I)} = I_0 = e^{\cdot(-I)} \cdot e^{\cdot I},$$

where $I_0 \in \operatorname{End} \wedge^0 V$ represents the unit element with respect to the shuffle product, one can therefore define an invertible map $\alpha$ of $\Pi_p \operatorname{End} \wedge^p V$ into itself by letting $\alpha\mathbf{A} \in \Pi_p \operatorname{End} \wedge^p V$ be the shuffle product $e^{\cdot I} \cdot \mathbf{A}$ for any $\mathbf{A} \in \Pi_p \operatorname{End} \wedge^p V$; the inverse $\alpha^{-1}$ of $\alpha$ is given by $\alpha^{-1}\mathbf{A} = e^{\cdot(-I)} \cdot \mathbf{A}$.

1.1 *Definition:* The *third product* of any two elements $\mathbf{A}$ and $\mathbf{B}$ of $\Pi_p$ End $\wedge^p V$ is given by $\mathbf{A} \times \mathbf{B} = \alpha^{-1}((\alpha\mathbf{A})(\alpha\mathbf{B})) \in \Pi_p$ End $\wedge^p V$, where $(\alpha\mathbf{A})(\alpha\mathbf{B})$ is the composition product of the shuffle products $\alpha\mathbf{A} = e^{\cdot I} \cdot \mathbf{A}$ and $\alpha\mathbf{B} = e^{\cdot I} \cdot \mathbf{B}$.

Since the composition product is associative the third product is trivially associative. Furthermore, if $I_0 \in$ End $\wedge^0 V$ represents the unit element in $\Pi_p$ End $\wedge^p V$ with respect to the shuffle product one has

$$I_0 \times \mathbf{A} = \alpha^{-1}((\alpha I_0)(\alpha\mathbf{A})) = \alpha^{-1}((e^{\cdot I})(\alpha\mathbf{A})) = \alpha^{-1}(\mathbf{I}(\alpha\mathbf{A})) = \alpha^{-1}(\alpha\mathbf{A}) = \mathbf{A}$$

and similarly $\mathbf{A} \times I_0 = \mathbf{A}$ for any $\mathbf{A} \in \Pi_p$ End $\wedge^p V$; that is, $I_0$ is also the unit element of $\Pi_p$ End $\wedge^p V$ with respect to the third product. The rationale for introducing the third product appears in the next section.

## 2. The trace

We now specialize the arbitrary $R$-module $V$ of the preceding section.

2.1 *Definition:* A module $V$ over a commutative ring $R$ with unit is *traceable* of rank $n > 0$ if and only if End $\wedge^n V$ is a free $R$-module of rank one.

If $\wedge^n V$ is itself free of rank one then $V$ is clearly traceable of rank $n$. However, End $\wedge^n V$ can be free of rank one with no such condition on $\wedge^n V$. For example, let $X$ be any paracompact hausdorff space, let $R$ be the ring $C(X)$ of continuous real-valued functions on $X$, and let $V$ be the $C(X)$-module of continuous sections of a real $n$-plane bundle $\xi$ over $X$; then $V$ is traceable of rank $n$. However $\wedge^n V$ is itself free of rank one if and only if $\xi$ is orientable.

Flanders [1] showed for any module $V$ over a commutative ring with unit that if $\wedge^n V$ is free of rank one then $\wedge^p V = 0$ for every $p > n$; a similar argument shows that if $V$ is traceable of rank $n > 0$ then End $\wedge^p V = 0$ for every $p > n$. Thus if $V$ is traceable of rank $n > 0$ there is no distinction between the direct product $\Pi_p$ End $\wedge^p V$ and the direct sum $\amalg_p$ End $\wedge^p V$. Consequently the third product of Definition 1.1 can be regarded as a product in $\amalg_p$ End $\wedge^p V$ whenever $V$ is traceable.

If $V$ is traceable of rank $n$ then every element of End $\wedge^n V$ is scalar multiplication by a unique element of the commutative ground ring $R$ with unit. For example, for any $\mathbf{A} \in \amalg_p$ End $\wedge^p V$ and each $p = 0, \ldots, n$ let