5. CONCLUDING REMARKS

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **32 (1986)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **26.05.2024**

$$n = 4^a 9^b, \quad n' = 2 \cdot 4^c 9^d (a, b, c, d \in \mathbf{N}),$$

for which

$$\lambda(n) = 1, \quad \lambda(n') = -1.$$

We therefore have obtained the desired contradiction under the assumption that there exist four consecutive integers $n \geqslant 2N_0$, for which $\lambda(n) = 1$. By the part of the theorem already proved, there exist at least three such integers. Therefore (5) holds for some $m_0 > 2N_0$ with $n_0 = m_0 + 2$, and we may now assume that

$$\lambda(m_0 - 1) = \lambda(m_0 + 3) = -1.$$

If m_0 is odd, then this implies

$$\lambda\left(\frac{m_0 - 1}{2}\right) = \lambda\left(\frac{m_0 + 3}{2}\right) = 1, \quad \lambda\left(\frac{m_0 + 1}{2}\right) = -1,$$

so that $(m_0 + 1)/2 > N_0$ is good, in contradiction to our assumption. But if m_0 is even, then defining m_1 and n_1 by (7), (6) holds for $i = 1$, and we have

$$m_1 \geqslant 2N_0, n_1 - m_1 = \frac{3(m_0 + 2)}{2} - \frac{3m_0}{2} = 3.$$

Thus we are back in the case already treated.

By contradiction, we therefore conclude that (1) has infinitely many solutions for $(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (1, -1, 1)$, and the proof of the theorem is complete.

5. Concluding remarks

In the foregoing proof, the relevant property of the Liouville function was that $\lambda(n)$ is completely multiplicative and assumes only the values ± 1. Besides this, we used only the fact that $\lambda(2) = \lambda(3) = \lambda(5) = -1$ and (in the proof of the lemma)

$$\lambda(14) = \lambda(16) = 1, \quad \lambda(29) = \lambda(31) = -1.$$

The proof, as it stands, works for any completely multiplicative function $f(n) = \pm 1$ with these properties. By suitably modifying the proof, it is possible to cover other classes of multiplicative functions as well.

It would be interesting to determine those completely multiplicative functions $f(n) = \pm 1$, for which the analogue of the theorem does not hold. Schur [3] proved that if $f \not\equiv f_{\pm}$, where

$$f_{\pm}(n) = \begin{cases} (\pm 1)^k & \text{if} \quad n = 3^k m, m \equiv 1 \bmod 3\,, \\ -(\pm 1)^k & \text{if} \quad n = 3^k m, m \equiv 2 \bmod 3\,, \end{cases}$$

then there exists at least one $n \geqslant 1$, such that

$$f(n) = f(n+1) = f(n+2) = 1\,.$$

It is likely that under the same hypotheses there are infinitely many such n. Using arguments similar to those in section 3, one can prove this assertion under the additional hypotheses $f(2) = 1$ and $f(3) = -1$, but the general case seems to be more complicated.

A very plausible conjecture is that the integers n, for which (1) holds, have positive density. In the case $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 1$, this would follow from an analogous strengthening of the lemma by requiring (2) to hold on a set of positive density. Whereas a very simple argument shows that the equations $\lambda(n) = \lambda(n+1)$ and $\lambda(n+1) = \lambda(n-1)$ hold on a set of positive (lower) density (cf. [2]), this argument seems to break down, if n is required to lie in a prescribed residue class, and so far we have not been able to overcome this difficulty.

REFERENCES

[1] Chowla, S. *The Riemann hypothesis and Hilbert's tenth problem.* New York-London-Paris, Gordon and Breach 1965.
[2] Graham, S. and D. Hensley. Problem E 3025. *Am. Math. Monthly 90* (1983), 707.
[3] Schur, I. Multiplikativ signierte Folgen positiver ganzer Zahlen. In: *Gesammelte Abhandlungen von Issai Schur, Vol. 3*, 392-399. Berlin-Heidelberg-New York, Springer 1973.

(Reçu le 21 octobre 1985)

Adolf Hildebrand

Department of Mathematics
University of Illinois
1409 West Green Street
Urbana, Illinois 61801
U.S.A.