

## **§2. Micro-differential Operators (See [SKK], [Bj], [S], [K2])**

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$\mathbb{C}[x_1, \dots, x_n]$	$\mathcal{D}_X$
$\mathbb{C}^n$	$T^*X$
the sheaf $\mathcal{O}_{\mathbb{C}^n}$ of holomorphic functions	$\mathcal{E}_X$

## § 2. MICRO-DIFFERENTIAL OPERATORS (See [SKK], [Bj], [S], [K2])

2.1. Let  $X$  be an  $n$ -dimensional complex manifold and let  $\pi_X: T^*X \rightarrow X$  be the cotangent bundle of  $X$ . Let us take a local coordinate system  $(x_1, \dots, x_n)$  of  $X$  and the associated coordinates  $(x_1, \dots, x_n; \xi_1, \dots, \xi_n)$  of  $T^*X$ . For a differential operator  $P$ , let  $\{P_j(x, \xi)\}$  be the total symbol of  $P$  as in § 1.2. We sometimes write  $P = \sum P_j(x, \partial)$ .

Let  $Q = \sum Q_j(x, \partial)$  be another differential operator. Set  $S = P + Q$  and  $R = PQ$ . Then the total symbols  $\{S_j\}$  and  $\{R_j\}$  of  $R$  and  $S$  are given explicitly by

$$(2.1.1) \quad S_j = P_j + Q_j$$

$$(2.1.2) \quad R_l = \sum_{\substack{l=j+k-|\alpha| \\ \alpha \in \mathbb{N}^n}} \frac{1}{\alpha!} (\partial_\xi^\alpha P_j)(\partial_x^\alpha Q_k)$$

where  $\partial_\xi^\alpha = (\partial/\partial\xi_1)^{\alpha_1} \dots (\partial/\partial\xi_n)^{\alpha_n}$  and  $\partial_x^\alpha = (\partial/\partial x_1)^{\alpha_1} \dots (\partial/\partial x_n)^{\alpha_n}$ .

The total symbol  $\{P_j(x, \xi)\}$  of a differential operator behaves as follows under coordinate transformations. Let  $(x_1, \dots, x_n)$  and  $(\tilde{x}_1, \dots, \tilde{x}_n)$  be two local coordinate systems. Let  $(\xi_1, \dots, \xi_n)$  and  $(\tilde{\xi}_1, \dots, \tilde{\xi}_n)$  be related by

$$\xi_k = \sum_j \tilde{\xi}_j \cdot \frac{\partial \tilde{x}_j}{\partial x_k}$$

i.e.  $(x_1, \dots, x_n; \xi_1, \dots, \xi_n)$  and  $(\tilde{x}_1, \dots, \tilde{x}_n; \tilde{\xi}_1, \dots, \tilde{\xi}_n)$  are the associated local coordinate systems of the cotangent bundle  $T^*X$ . Let  $P$  be a differential operator on  $X$  and let  $\{P_j(x, \xi)\}$  and  $\{\tilde{P}_j(\tilde{x}, \tilde{\xi})\}$  be the total symbols of  $P$  with respect to the local coordinate systems  $(x_1, \dots, x_n)$  and  $(\tilde{x}_1, \dots, \tilde{x}_n)$ , respectively. Then one has

$$(2.1.3) \quad \begin{aligned} & \tilde{P}_l(\tilde{x}, \tilde{\xi}) \\ &= \sum_{v, \alpha_1, \dots, \alpha_v} \frac{1}{v! \alpha_1! \dots \alpha_v!} \langle \tilde{\xi}, \partial_x^{\alpha_1} \tilde{x} \rangle \dots \langle \tilde{\xi}, \partial_x^{\alpha_v} \tilde{x} \rangle \partial_\xi^{\alpha_1 + \dots + \alpha_v} P_j(x, \xi). \end{aligned}$$

Here the indices run over  $j \in \mathbf{Z}$ ,  $v \in \mathbf{N}$ ,  $\alpha_1, \dots, \alpha_v \in \mathbf{N}^n$  such that  $|\alpha_1|, \dots, |\alpha_v| \geq 2$  and  $l = j + v - |\alpha_1| - \dots - |\alpha_v|$ . For  $\beta \in \mathbf{N}^n$ ,  $\langle \tilde{\xi}, \partial_x^\beta \tilde{x} \rangle$  denotes  $\sum_j \tilde{\xi}_j \partial_x^\beta \tilde{x}_j$ .

2.2. The total symbol  $\{P_j(x, \xi)\}$  of a differential operator is a polynomial in  $\xi$ . We shall define microdifferential operators by admitting  $P_j$  to be holomorphic in  $\xi$ .

For  $\lambda \in \mathbf{C}$ , let  $\mathcal{O}_{T^*X}(\lambda)$  be the sheaf of homogeneous holomorphic functions of degree  $\lambda$  on  $T^*X$ , i.e., holomorphic functions  $f(x, \xi)$  satisfying

$$(\sum \xi_j \partial/\partial \xi_j - \lambda) f(x, \xi) = 0.$$

*Definition 2.2.1.* For  $\lambda \in \mathbf{C}$  we define the sheaf  $\mathcal{E}_X(\lambda)$  on  $T^*X$  by

$$\Omega \mapsto \{(P_{\lambda-j}(x, \xi))_{j \in \mathbf{N}} ; P_{\lambda-j} \in \Gamma(\Omega; \mathcal{O}_{T^*X}(\lambda-j)) \text{ and satisfies the following conditions (2.2.1)}\}$$

(2.2.1) for any compact subset  $K$  of  $\Omega$ , there exists a  $C_K > 0$  such that

$$\sup_K |P_{\lambda-j}| \leq C_K^{-j} (j!) \quad \text{for all } j > 0.$$

*Remark.* The growth condition (2.2.1) can be explained as follows. For a differential operator  $P = \sum P_j(x, \partial)$ , we have

$$P(x, \partial) (\langle x, \xi \rangle + p)^\mu = \sum P_j(x, \xi) \frac{\Gamma(\mu)}{\Gamma(\mu-j+1)} (\langle x, \xi \rangle + p)^{\mu-j}.$$

For  $P = (P_{\lambda-j}(x, \xi)) \in \mathcal{E}(\lambda)$  we set, by analogy

$$P(\langle x, \xi \rangle + p)^\mu = \sum_j P_{\lambda-j}(x, \xi) \frac{\Gamma(\mu)}{\Gamma(\mu-\lambda+j+1)} (\langle x, \xi \rangle + p)^{\mu-\lambda+j}.$$

Then the growth condition (2.2.1) is simply the condition that the right hand side converges when  $0 < |\langle x, \xi \rangle + p| \ll 1$ .

Now, we have the following

PROPOSITION 2.2.2 ([SKK], Chap. II, § 1, [Bj] Chap. IV, § 1).

- (0)  $\mathcal{E}_X(\lambda)$  contains  $\mathcal{E}_X(\lambda-m)$  as a subsheaf for  $m \in \mathbf{N}$ .
- (1) Patching by rule (2.1.3) under coordinate transformations,  $\mathcal{E}_X(\lambda)$  becomes a sheaf defined globally on  $T^*X$ .

- (2) By rule (2.1.1),  $\mathcal{E}_X(\lambda)$  is a sheaf of  $\mathbf{C}$ -vector space on  $T^*X$ .  
(3) By rule (2.1.2), we can define the “product” homomorphism:

$$\mathcal{E}_X(\lambda) \underset{\mathbf{C}}{\otimes} \mathcal{E}_X(\mu) \rightarrow \mathcal{E}_X(\lambda + \mu),$$

which satisfies the associative law.

- (4) In particular,  $\mathcal{E}_X(0)$  and  $\mathcal{E}_X = \bigcup_{m \in \mathbf{Z}} \mathcal{E}_X(m)$  become sheaves of (non commutative) rings on  $T^*X$ , with a unit.

The unit is given by  $(P_j(x, \xi))$  with  $P_j = 1$  for  $j = 0$  and  $P_j = 0$  for  $j \neq 0$ .

We define the homomorphism

$$\sigma_\lambda: \mathcal{E}_X(\lambda) \rightarrow \mathcal{O}_{T^*X}(\lambda)$$

by

$$(P_{\lambda-j}) \mapsto P_\lambda.$$

Then,  $\sigma_\lambda$  is a well-defined homomorphism on  $T^*X$  (i.e. compatible with coordinate transformation) and we have an exact sequence

$$0 \rightarrow \mathcal{E}_X(\lambda - 1) \rightarrow \mathcal{E}_X(\lambda) \xrightarrow{\sigma_\lambda} \mathcal{O}_{T^*X}(\lambda) \rightarrow 0.$$

Now we have the following proposition, which says that the ring  $\mathcal{E}_X$  is a kind of localization of  $\mathcal{D}_X$ .

### PROPOSITION 2.2.3.

- (1) For  $P \in \mathcal{E}(\lambda)$  and  $Q \in \mathcal{E}(\mu)$ , we have  $\sigma_{\lambda+\mu}(PQ) = \sigma_\lambda(P)\sigma_\mu(Q)$ .  
(2) ([SKK] Chap. II, Thm. 2.1.1) If  $P \in \mathcal{E}(\lambda)$  satisfies  $\sigma_\lambda(P)(q) \neq 0$  at  $q \in T^*X$ , then there exists  $Q \in \mathcal{E}(-\lambda)$  such that  $PQ = QP = 1$ .

The relations between  $\mathcal{E}_X$  and  $\mathcal{D}_X$  are summarized in the following theorem.

### THEOREM 2.2.4 ([SKK], Chap. II, § 3).

- (i)  $\mathcal{E}_X$  contains  $\pi^{-1}\mathcal{D}_X$  as a subring and is flat over  $\pi^{-1}\mathcal{D}_X$ .  
(ii)  $\mathcal{E}_X|_{T_X^*X} \simeq \mathcal{D}_X$ , where  $T_X^*X$  is the zero section of  $T^*X$ .  
(iii) For a coherent  $\mathcal{D}_X$ -module  $\mathcal{M}$ , the characteristic variety of  $\mathcal{M}$  coincides with the support of  $\mathcal{E}_X \underset{\pi_X^{-1}\mathcal{D}_X}{\otimes} \pi_X^{-1}\mathcal{M}$ .