§2. Micro-differential Operators (See [SKK], [Bj], [S], [K2])

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$\mathbf{C}[x_1, \ldots, x_n]$	$\mathscr{D}_X$
$\mathbf{C}^n$	T^*X
the sheaf $\mathscr{O}_{\mathbf{C}^n}$ of holomorphic functions	$\mathscr{E}_X$

§ 2. Micro-differential Operators (See [SKK], [Bj], [S], [K2])

2.1. Let X be an n-dimensional complex manifold and let $\pi_X : T^*X \to X$ be the cotangent bundle of X. Let us take a local coordinate system $(x_1, \ldots, x_n)$ of X and the associated coordinates $(x_1, \ldots, x_n; \xi_1, \ldots, \xi_n)$ of T^*X. For a differential operator P, let $\{P_j(x, \xi)\}$ be the total symbol of P as in § 1.2. We sometimes write $P = \Sigma P_j(x, \partial)$.

Let $Q = \Sigma Q_j(x, \partial)$ be another differential operator. Set $S = P + Q$ and $R = PQ$. Then the total symbols $\{S_j\}$ and $\{R_j\}$ of R and S are given explicitly by

$$S_j = P_j + Q_j \tag{2.1.1}$$

$$R_l = \sum_{\substack{l = j+k-|\alpha| \\ \alpha \in \mathbf{N}^n}} \frac{1}{\alpha\,!} (\partial_\xi^\alpha P_j)(\partial_x^\alpha Q_k) \tag{2.1.2}$$

where $\partial_\xi^\alpha = (\partial/\partial\xi_1)^{\alpha_1} \ldots (\partial/\partial\xi_n)^{\alpha_n}$ and $\partial_x^\alpha = (\partial/\partial x_1)^{\alpha_1} \ldots (\partial/\partial x_n)^{\alpha_n}$.

The total symbol $\{P_j(x, \xi)\}$ of a differential operator behaves as follows under coordinate transformations. Let $(x_1, \ldots, x_n)$ and $(\tilde{x}_1, \ldots, \tilde{x}_n)$ be two local coordinate systems. Let $(\xi_1, \ldots, \xi_n)$ and $(\tilde{\xi}_1, \ldots, \tilde{\xi}_n)$ be related by

$$\xi_k = \sum_j \tilde{\xi}_j \cdot \frac{\partial \tilde{x}_j}{\partial x_k}$$

i.e. $(x_1, \ldots, x_n; \xi_1, \ldots, \xi_n)$ and $(\tilde{x}_1, \ldots, \tilde{x}_n; \tilde{\xi}_1, \ldots, \tilde{\xi}_n)$ are the associated local coordinate systems of the cotangent bundle T^*X. Let P be a differential operator on X and let $\{P_j(x, \xi)\}$ and $\{\tilde{P}_j(\tilde{x}, \tilde{\xi})\}$ be the total symbols of P with respect to the local coordinate systems $(x_1, \ldots, x_n)$ and $(\tilde{x}_1, \ldots, \tilde{x}_n)$, respectively. Then one has

$$\begin{aligned} &\tilde{P}_l(\tilde{x}, \tilde{\xi}) \\ &= \sum_{\nu, \alpha_1, \ldots, \alpha_\nu} \frac{1}{\nu\,!\,\alpha_1\,! \ldots \alpha_\nu\,!} \langle \tilde{\xi}, \partial_x^{\alpha_1} \tilde{x} \rangle \ldots \langle \tilde{\xi}, \partial_x^{\alpha_\nu} \tilde{x} \rangle \, \partial_\xi^{\alpha_1 + \cdots + \alpha_\nu} P_j(x, \xi)\,. \end{aligned} \tag{2.1.3}$$

Here the indices run over $j \in \mathbf{Z}$, $\nu \in \mathbf{N}$, $\alpha_1, ..., \alpha_\nu \in \mathbf{N}^n$ such that $|\alpha_1|, ..., |\alpha_\nu| \geqslant 2$ and $l = j + \nu - |\alpha_1| - ... - |\alpha_\nu|$. For $\beta \in \mathbf{N}^n$, $<\tilde{\xi}, \partial_x^\beta \tilde{x}>$ denotes $\sum_j \xi_j \partial^\beta \tilde{x}_j$.

2.2. The total symbol $\{P_j(x, \xi)\}$ of a differential operator is a polynomial in ξ. We shall define microdifferential operators by admitting P_j to be holomorphic in ξ.

For $\lambda \in \mathbf{C}$, let $\mathcal{O}_{T*X}(\lambda)$ be the sheaf of homogeneous holomorphic functions of degree λ on T^*X, i.e., holomorphic functions $f(x, \xi)$ satisfying

$$(\Sigma \xi_j \partial/\partial \xi_j - \lambda) f(x, \xi) = 0 .$$

Definition 2.2.1. For $\lambda \in \mathbf{C}$ we define the sheaf $\mathscr{E}_X(\lambda)$ on T^*X by

$$\Omega \mapsto \{(P_{\lambda-j}(x, \xi))_{j \in \mathbf{N}} ; \quad P_{\lambda-j} \in \Gamma(\Omega; \mathcal{O}_{T*X}(\lambda - j))$$
$$\text{and satisfies the following conditions (2.2.1)}\}$$

(2.2.1) for any compact subset K of Ω, there exists a $C_K > 0$ such that

$$\sup_K |P_{\lambda-j}| \leqslant C_K^{-j}(j!) \quad \text{for all} \quad j > 0 .$$

Remark. The growth condition (2.2.1) can be explained as follows. For a differential operator $P = \Sigma P_j(x, \partial)$, we have

$$P(x, \partial)(<x, \xi> + p)^\mu = \Sigma P_j(x, \xi) \frac{\Gamma(\mu)}{\Gamma(\mu - j + 1)} (<x, \xi> + p)^{\mu - j} .$$

For $P = (P_{\lambda-j}(x, \xi)) \in \mathscr{E}(\lambda)$ we set, by analogy

$$P(<x, \xi> + p)^\mu = \sum_j P_{\lambda-j}(x, \xi) \frac{\Gamma(\mu)}{\Gamma(\mu - \lambda + j + 1)} (<x, \xi> + p)^{\mu - \lambda + j} .$$

Then the growth condition (2.2.1) is simply the condition that the right hand side converges when $0 < |<x, \xi> + p| \ll 1$.

Now, we have the following

Proposition 2.2.2 ([SKK], Chap. II, § 1, [Bj] Chap. IV, § 1).

(0) *$\mathscr{E}_X(\lambda)$ contains $\mathscr{E}_X(\lambda - m)$ as a subsheaf for* $m \in \mathbf{N}$.

(1) *Patching by rule* (2.1.3) *under coordinate transformations, $\mathscr{E}_X(\lambda)$ becomes a sheaf defined globally on T^*X.*

(2) *By rule* (2.1.1), $\mathscr{E}_X(\lambda)$ *is a sheaf of* **C**-*vector space on* T^*X.

(3) *By rule* (2.1.2), *we can define the* "product" *homomorphism:*

$$\mathscr{E}_X(\lambda) \underset{\mathbf{C}}{\otimes} \mathscr{E}_X(\mu) \to \mathscr{E}_X(\lambda+\mu),$$

which satisfies the associative law.

(4) *In particular,* $\mathscr{E}_X(0)$ *and* $\mathscr{E}_X = \bigcup_{m\in\mathbf{Z}} \mathscr{E}_X(m)$ *become sheaves of (non commutative) rings on* T^*X, *with a unit.*

The unit is given by $(P_j(x, \xi))$ *with* $P_j = 1$ *for* $j = 0$ *and* $P_j = 0$ *for* $j \neq 0$.

We define the homomorphism

$$\sigma_\lambda : \mathscr{E}_X(\lambda) \to \mathcal{O}_{T^*X}(\lambda)$$

by

$$(P_{\lambda-j}) \mapsto P_\lambda .$$

Then, σ_λ is a well-defined homomorphism on T^*X (i.e. compatible with coordinate transformation) and we have an exact sequence

$$0 \to \mathscr{E}_X(\lambda-1) \to \mathscr{E}_X(\lambda) \overset{\sigma_\lambda}{\to} \mathcal{O}_{T^*X}(\lambda) \to 0 .$$

Now we have the following proposition, which says that the ring $\mathscr{E}_X$ is a kind of localization of $\mathscr{D}_X$.

PROPOSITION 2.2.3.

(1) *For* $P \in \mathscr{E}(\lambda)$ *and* $Q \in \mathscr{E}(\mu)$, *we have* $\sigma_{\lambda+\mu}(PQ) = \sigma_\lambda(P)\sigma_\mu(Q)$.

(2) ([SKK] Chap. II, Thm. 2.1.1) *If* $P \in \mathscr{E}(\lambda)$ *satisfies* $\sigma_\lambda(P)(q) \neq 0$ *at* $q \in T^*X$, *then there exists* $Q \in \mathscr{E}(-\lambda)$ *such that* $PQ = QP = 1$.

The relations between $\mathscr{E}_X$ and $\mathscr{D}_X$ are summarized in the following theorem.

THEOREM 2.2.4 ([SKK], Chap. II, § 3).

(i) $\mathscr{E}_X$ *contains* $\pi^{-1}\mathscr{D}_X$ *as a subring and is flat over* $\pi^{-1}\mathscr{D}_X$.

(ii) $\mathscr{E}_X|_{T^*_XX} \simeq \mathscr{D}_X$, *where* T^*_XX *is the zero section of* T^*X.

(iii) *For a coherent* $\mathscr{D}_X$-*module* $\mathscr{M}$, *the characteristic variety of* $\mathscr{M}$ *coincides with the support of* $\mathscr{E}_X \underset{\pi_X^{-1}\mathscr{D}_X}{\otimes} \pi_X^{-1}\mathscr{M}$.