§5. The Vanishing Cycle Sheaf

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§ 5. The Vanishing Cycle Sheaf

5.1. Let M be a real manifold and $f: M \to \mathbf{R}$ a continuous map. For a sheaf $\mathscr{F}$ on M, $\mathscr{H}^j_{f^{-1}(\mathbf{R}^+)}(\mathscr{F})\,|_{f^{-1}(0)}$ is called the (j-th) *vanishing cycle sheaf* of $\mathscr{F}$. Here $\mathbf{R}^+ = \{t \in \mathbf{R}; t \geqslant 0\}$. This measures how the cohomology groups of $\mathscr{F}$ change across the fibers of f. Its algebro-geometric version is studied by Grothendieck-Deligne ([D]).

5.2. Let $(X, \mathscr{O}_X)$ be a complex manifold. Let $f: X \to \mathbf{R}$ be a C^∞-map and consider the vanishing cycle sheaf $\mathscr{H}^j_{f^{-1}(\mathbf{R}^+)}(\mathscr{O}_X)\,|_{f^{-1}(0)}$. Let s be the section of $f^{-1}(0) \to T^*X$ given by df. Then we have

Proposition 5.2.1 ([KS1] § 3, [K2] § 4.2). *$\mathscr{H}^j_{f^{-1}(\mathbf{R}^+)}(\mathscr{O}_X)\,|_{f^{-1}(0)}$ has a structure of an $s^{-1}\mathscr{E}_X$-module.*

Let P be a differential operator. If $\sigma(P)$ does not vanish on $s(f^{-1}(0))$, then P has an inverse in $s^{-1}\mathscr{E}_X$ by Proposition 2.2.3. Therefore we obtain

Corollary 5.2.2. *If $\sigma(P)\,|_{sf^{-1}(0)} \neq 0$, then*

$$P: \mathscr{H}^j_{f^{-1}\mathbf{R}^+}(\mathscr{O}_X)\,|_{f^{-1}(0)} \to \mathscr{H}^j_{f^{-1}(\mathbf{R}^+)}(\mathscr{O}_X)\,|_{f^{-1}(0)}$$

is bijective.

5.3. More generally, let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module, and set

$$\mathscr{F}^\bullet = \mathbf{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X).$$

Then the preceding corollary shows that

$$\mathbf{R}\Gamma_{f^{-1}\mathbf{R}^+}(\mathscr{F}^\bullet)\,|_{f^{-1}(0)} = 0 \quad \text{if} \quad s(f^{-1}(0)) \cap \mathrm{Ch}(\mathscr{M}) = \emptyset.$$

Here $\mathrm{Ch}\mathscr{M}$ denotes the characteristic variety of $\mathscr{M}$.

5.4. To consider vanishing cycle sheaves is very near to the "microlocal" consideration. In this direction, see [K-S2].

§ 6. Micro-Differential Operators and the Symplectic Structure on the Cotangent Bundle

6.1. The ring $\mathscr{E}_X$ is a non-commutative ring. This fact gives rise to new phenomena which are not shared by commutative rings such as the ring of