# §11. Application to the b-function (see [SKKO])

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*is an isomorphism.*

*In particular if* $\operatorname{Supp}\mathcal{M} \subset \Lambda_1 \cup \Lambda_2$ *and if* $\dim(\Lambda_1 \cap \Lambda_2) \leqslant n-2$, *then* $\mathcal{M}$ *is a direct sum of two holonomic* $\mathcal{E}_X$*-modules supported on* $\Lambda_1$ *and* $\Lambda_2$, *respectively.*

Here is another type of theorem.

THEOREM 10.4.3 ([SKKO]). *Let* $\mathcal{M} = \mathcal{E}u = \mathcal{E}/\mathcal{J}$ *be a holonomic* $\mathcal{E}$*-module defined on a neighborhood of* $p \in T^*X$. *Assume* $\operatorname{Supp}\mathcal{M} = \Lambda_1 \cup \Lambda_2$ *and*

(i) $\Lambda_1, \Lambda_2$ *and* $\Lambda_1 \cap \Lambda_2$ *are non-singular and* $\dim \Lambda_1 = \dim \Lambda_2 = n$, $\dim(\Lambda_1 \cap \Lambda_2) = n-1$.

(ii) $T_{p'}\Lambda_1 \cap T_{p'}\Lambda_2 = T_{p'}(\Lambda_1 \cap \Lambda_2)$ *for any* $p'$ *in a neighborhood of* $p$ *in* $\Lambda_1 \cap \Lambda_2$.

(iii) *The symbol ideal of* $\mathcal{J}$ *coincides with the ideal of functions vanishing on* $\Lambda_1 \cup \Lambda_2$.

*Setting* $k = \operatorname{ord}_{\Lambda_1}u - \operatorname{ord}_{\Lambda_2}u - 1/2$, *we have*

(a) $\mathcal{M}$ *has a non-zero quotient supported on* $\Lambda_1 \Leftrightarrow \mathcal{M}$ *has a non-zero submodule supported on* $\Lambda_2 \Leftrightarrow k \in \mathbf{Z}$.

(b) $\mathcal{M}_p$ *is a simple* $\mathcal{E}_p$*-module* $\Leftrightarrow k \notin \mathbf{Z}$.

*Sketch of the proof.* By a quantized contact transformation, we can transform $p$, $\Lambda_1$, $\Lambda_2$ and $\mathcal{J}$ as follows:

$$p = (0, dx_1)$$

$$\Lambda_1 = \{(x,\xi);\, x_1 = \xi_2 = \dots = \xi_n = 0\}$$

$$\Lambda_2 = \{(x,\xi);\, x_1 = x_2 = \xi_3 = \dots = \xi_n = 0\}$$

$$\mathcal{J} = \mathcal{E}(x_1\partial_1 - \lambda) + \mathcal{E}(x_2\partial_2 - \mu) + \sum_{j>2} \mathcal{E}\partial_j$$

In this case, we can easily check the theorem.

## § 11. APPLICATION TO THE $b$-FUNCTION (see [SKKO])

11.1. As one of the most successful application of microlocal analysis, we shall sketch here how to calculate the $b$-function of a function under certain conditions.

11.2. Let $f$ be a holomorphic function on a complex manifold $X$. Then, it is proved ([Bj], [Be] [K1]) that there exist (locally) a non zero polynomial $b(s)$ and $P(s) \in \mathscr{D}[s] \underset{\text{def}}{=} \mathscr{D} \underset{\mathbf{C}}{\otimes} \mathbf{C}[s]$ such that $P(s)f(x)^{s+1} = b(s)f(x)^s$ for any $s \in \mathbf{N}$. Such a polynomial $b(s)$ of smallest degree is called the $b$-function of $f(x)$ and is denoted by $b_f(s)$. For the relations between the $b$-function and the local monodromy see [M1], [K3].

11.3. Set $\mathscr{J} = \{P(s) \in \mathscr{D}[s]; P(s)f^s = 0 \text{ for } s \in \mathbf{N}\}$ and $\mathscr{N} = \mathscr{D}[s]/\mathscr{J}$. We shall denote the canonical generator of $\mathscr{N}$ by $f^s$. Then $t: \mathscr{N} \ni P(s)f^s \to P(s+1)f \cdot f^s \in \mathscr{N}$ gives a $\mathscr{D}$-endomorphism of $\mathscr{N}$ and $t\mathscr{N} = \mathscr{D}[s]f^{s+1}$. Here $f^{s+1} = f \cdot f^s \in \mathscr{N}$. In this terminology $b_f(s)$ is the minimal polynomial of $s \in \mathscr{E}nd_{\mathscr{D}}(\mathscr{N}/t\mathscr{N})$.

For $\lambda \in \mathbf{C}$, we set $\mathscr{M}_\lambda = \mathscr{D}[s]/(\mathscr{J} + \mathscr{D}[s](s-\lambda))$ and denote by $f^\lambda$ the canonical generator of $\mathscr{M}_\lambda$. Then $f^{\lambda+1} \mapsto f\, f^\lambda$ defines a $\mathscr{D}$-linear homomorphism $\mathscr{M}_{\lambda+1} \to \mathscr{M}_\lambda$.

11.4. Let $W$ be the closure of

$$\{(s, x, \xi) \in \mathbf{C} \times T^*X; \xi = s d \log f(x), f(x) \neq 0\}$$

in $\mathbf{C} \times T^*X$. Set $W_0 = W \cap \{s=0\} \subset T^*X$. Then we can prove

PROPOSITION 11.4.1 ([K1]).

(i) *$N$ is a coherent $\mathscr{D}_X$-module and* $\mathrm{Ch}(\mathscr{N}) = p(W)$, *where $p$ is the projection from* $\mathbf{C} \times T^*X$ *to* $T^*X$.

(ii) *For any* $\lambda \in \mathbf{C}$, $\mathscr{M}_\lambda$ *is a regular holonomic $\mathscr{D}_X$-module and* $\mathrm{Ch}(\mathscr{M}_\lambda) = W_0$.

(iii) $\mathscr{N}/t\mathscr{N}$ *is a regular holonomic $\mathscr{D}_X$-module and* $\mathrm{Ch}(\mathscr{N}/t\mathscr{N}) = W_0 \cap (\pi \circ f)^{-1}(0)$.

11.5. In the sequel, for the sake of simplicity, we assume that there exists a vector field $v$ such that $v(f) = f$. Therefore we have $v^k(f^s) = s^k f^s$. Hence $\mathscr{N}$ is a $\mathscr{D}$-module generated by $f^s$. If we set $\tilde{\mathscr{J}} = \mathscr{D} \cap \mathscr{J}$, then $\mathscr{N} \cong \mathscr{D}/\mathscr{J}$ and $\mathscr{J} = \mathscr{D}[s](s-v) + \mathscr{D}[s]\tilde{\mathscr{J}}$.

11.6. The following lemma is almost obvious but affords a fundamental tool to calculate the $b$-function.

LEMMA 11.6.1. *Let $\mathscr{L}$ be an $\mathscr{E}_X$-module and $w$ a non-zero section of $\mathscr{L}$. For $\lambda \in \mathbf{C}$, we assume*

(i) $v(w) = \lambda w$

(ii) $\tilde{\mathscr{J}} w = 0$

(iii) $fw = 0$.

*Then we have* $b_f(\lambda) = 0$.

*Proof.* There is a $P \in \mathscr{D}$ such that $b_f(s)f^s = Pf^{s+1}$. Hence $(b_f(v)-Pf)f^s = 0$, which implies $b_f(v) - Pf \in \tilde{\mathscr{J}}$. Since $b_f(v)w = b_f(\lambda)w$ we have

$$0 = (b_f(v)-Pf)w = b_f(\lambda)w\,.$$

This implies $b_f(\lambda) = 0$.

11.7. Let $\bar{\mathscr{J}}$ be the symbol ideal of $\tilde{\mathscr{J}}$. Then the zero set of $\bar{\mathscr{J}}$ is $W$, and the zero of $\bar{\mathscr{J}} + \mathscr{O}\sigma(v)$ is $W_0$. Let $\Lambda$ be an irreducible component of $W_0$. If $\bar{\mathscr{J}} + \mathscr{O}_{T*X}\sigma(v)$ is a reduced ideal at a generic point $p$ of $\Lambda$ then we call $\Lambda$ a *good Lagrangean.*

If $\Lambda$ is a good Lagrangean, then $W$ is non-singular on a neighborhood of a generic point $p$ of $\Lambda$ and $\sigma = \sigma(s)|_W$ has non zero-differential. Let $p: W \to X$ denote the projection. We define $m(\Lambda) \in \mathbf{N}$ as the degree of zero of $f \circ p$ along $\Lambda$, and set $f_\Lambda = (f \circ p/\sigma^{m(\Lambda)})|_\Lambda$. Let $\omega$ be the non-vanishing $n$-form on $X$. Then $(p^*\omega) \wedge d\sigma$ is an $(n+1)$-form on $W$. Let $\mu(\Lambda)$ be the degree of zeros of $(p^*\omega) \wedge d\sigma$ along $\Lambda$, and let $\eta$ be the $n$-form on $\Lambda$ given by

$$\left.\frac{p^*\omega \wedge d\sigma}{\sigma^{\mu(\Lambda)}}\right|_\Lambda = \eta \wedge d\sigma\,.$$

If we set $\kappa_\Lambda = \eta \otimes \omega^{\otimes -1} \in \omega_\Lambda \otimes \omega_X^{\otimes -1}$, then this is independent of the choice of $\omega$. We have

PROPOSITION 11.7.1 ([SKKO]). *If $\Lambda$ is a good Lagrangean, then for any $\lambda \in \mathbf{C}$, $\mathscr{M}_\lambda$ is a simple holonomic system on a neighborhood of a generic point $p$ of $\Lambda$ and we have*

(i) $\sigma(f^\lambda) = f_\Lambda^\lambda \sqrt{\kappa_\Lambda}\,.$

*In particular*

$$\operatorname{ord} f^\lambda = -m(\Lambda)\lambda - \mu(\Lambda)/2\,.$$

(ii) *There exists a monic polynomial* $b_\Lambda(s)$ *of degree* $m(\Lambda)$ *and an invertible micro-differential operator* $P_\Lambda$ *of order* $m(\Lambda)$ *such that*

$$b_\Lambda(s)f^s = P_\Lambda f \cdot f^s \quad \textit{in} \quad \mathscr{E} \underset{\mathscr{D}}{\otimes} \mathscr{N}$$

*and* $$\sigma(P_\Lambda)|_\Lambda = f_\Lambda^{-1}.$$

Remark that $f_\Lambda$ and $\omega_\Lambda$ are homogeneous of degree $-m(\Lambda)$ and $-\mu(\Lambda)$, respectively.

Remark also that the minimal polynomial of $s \in \mathscr{E}nd_{\mathscr{E}}(\mathscr{E} \underset{D}{\otimes} \mathscr{N}/t\mathscr{N})|_\Lambda$ is $b_\Lambda(s)$. In fact, if $Pf^{s+1} = b(s)f^s$ in $\mathscr{E} \otimes \mathscr{N}$, then $(P \cdot P_\Lambda^{-1}b_\Lambda(s) - b(s))f^s = 0$. This implies that $P \cdot P_\Lambda^{-1}b_\Lambda(v) - b(v) \in \mathscr{E}\tilde{\mathscr{J}}$. Hence

$$\sigma(P \cdot P_\Lambda^{-1}b_\Lambda(v) - b(v))|_W = 0.$$

If $\operatorname{ord} P \cdot P_\Lambda^{-1}b_\Lambda(v) = \operatorname{ord} P > \deg b$, then $\sigma(P)|_W = 0$. Therefore $P = P' + P''$ with $P'' \in \mathscr{E}\tilde{\mathscr{J}}$ and $\sigma(P') < \sigma(P)$. Hence $P'f^{s+1} = b(s)f^s$. Thus, we may assume $\operatorname{ord} P \leqslant \deg b$. Then

$$0 = \sigma(b(v) - P \cdot P_\Lambda^{-1}b_\Lambda(v))|_W = b(\sigma) - (\sigma(P)|_W f_\Lambda b_\Lambda(\sigma)).$$

This shows that $b(s)$ is a multiple of $b_\Lambda(s)$.

COROLLARY 11.7.2. *If every irreducible component of* $W_0$ *is good Lagrangean, then* $b_f(s)$ *is the least common multiple of the* $b_\Lambda(s)$.

11.8. Let $\Lambda_1$ and $\Lambda_2$ be two good Lagrangeans. We assume the following conditions for a point $p \in \Lambda_1 \cap \Lambda_2$:

(11.8.1) $\dim_p \Lambda_1 \cap \Lambda_2 = n-1$ and $\Lambda_1$, $\Lambda_2$ and $\Lambda_1 \cap \Lambda_2$ are non singular on a neighborhood of $p$.

(11.8.2) For any point $p'$ on a neighborhood of $p$ in $\Lambda_1 \cap \Lambda_2$, we have $T_{p'}\Lambda_1 \cap T_{p'}\Lambda_2 = T_{p'}(\Lambda_1 \cap \Lambda_2)$.

(11.8.3) $\bar{\mathscr{J}} + \mathscr{O}\sigma(v)$ coincides with the defining ideal of $\Lambda_1 \cup \Lambda_2$ with the reduced structure.

In this case we say that $\Lambda_1$ and $\Lambda_2$ have a *good intersection*.

We have the following theorem.

THEOREM 11.7.3. *Let* $\Lambda_1$ *and* $\Lambda_2$ *be good Lagrangeans with a good intersection. If* $m(\Lambda_1) \geqslant m(\Lambda_2)$, *then*

$$\prod_{k=0}^{m(\Lambda_1)-m(\Lambda_2)-1}\left(\operatorname{ord}_{\Lambda_2} f^s - \operatorname{ord}_{\Lambda_1} f^s + \frac{1}{2} + k\right) \mid b_f(s).$$

In order to prove this let us take $\lambda \in \mathbf{C}$ such that

$$(11.8.4) \qquad \begin{aligned} k &= \operatorname{ord}_{\Lambda_1} f^{\lambda} - \operatorname{ord}_{\Lambda_2} f^{\lambda} - 1/2 \in \mathbf{N} \quad \text{and} \\ k' &= \operatorname{ord}_{\Lambda_1} f^{\lambda+1} - \operatorname{ord}_{\Lambda_2} f^{\lambda+1} - 1/2 \in \mathbf{N}. \end{aligned}$$

Recall that

$$k = (m(\Lambda_2)-m(\Lambda_1))\lambda - \frac{1}{2}(\mu(\Lambda_2)-\mu(\Lambda_1)-1/2)$$

and $k' = k + (m(\Lambda_2)-m(\Lambda_1))$. Then by Theorem 10.4.3, $\mathscr{M}_\lambda$ has a non-zero quotient $\mathscr{L}$ whose support is $\Lambda_1$. Let $w \in \mathscr{L}$ be the image of $f^\lambda \in \mathscr{M}_\lambda$.

Let $\alpha: \mathscr{M}_\lambda \to \mathscr{L}$ be the canonical homomorphism and $\beta: \mathscr{M}_{\lambda+1} \to \mathscr{M}_\lambda$ be the homomorphism given by $f^{\lambda+1} \mapsto f \cdot f^\lambda$. Then, since $k' \notin \mathbf{N}$, $\mathscr{M}_{\lambda+1}$ has no non-zero quotient supported in $\Lambda_1$. Hence $\alpha \circ \beta = 0$. Therefore $fw = \alpha\beta(f^{\lambda+1}) = 0$. Thus we can apply Lemma 11.6.1 to conclude that $b_f(\lambda) = 0$. If $k \in \mathbf{Z}$ with $0 \leqslant k < m(\Lambda_1) - m(\Lambda_2)$ then

$$\lambda = \frac{1}{m(\Lambda_1) - m(\Lambda_2)}\left(k + \frac{1}{2}(\mu(\Lambda_1)-\mu(\Lambda_2)-1)\right)$$

satisfies (11.8.4). This shows that $b_f(s)$ is a multiple of

$$\begin{aligned} &\prod_{k=0}^{m(\Lambda_1)-m(\Lambda_2)-1}\left((m(\Lambda_1)-m(\Lambda_2))s - \frac{1}{2}(\mu(\Lambda_1)-\mu(\Lambda_2)-1) + k\right) \\ &= \text{const.} \prod_{k=0}^{m(\Lambda_1)-m(\Lambda_2)-1}\left(\operatorname{ord}_{\Lambda_2} f^s - \operatorname{ord}_{\Lambda_1} f^s + \frac{1}{2} + k\right). \end{aligned}$$

If we refine this argument, we can prove

THEOREM 11.8.2 ([SKKO]). *If* $\Lambda_1$ *and* $\Lambda_2$ *are good Lagrangeans with a good intersection and if* $m(\Lambda_1) \geqslant m(\Lambda_2)$ *then*

$$\frac{b_{\Lambda_1}(s)}{b_{\Lambda_2}(s)} = \text{const.} \prod_{k=0}^{m(\Lambda_1)-m(\Lambda_2)-1}\left(\operatorname{ord}_{\Lambda_2} f^s - \operatorname{ord}_{\Lambda_1} f^s + \frac{1}{2} + k\right).$$

*Example 11.8.3.*

(i) $X = M_n(\mathbf{C}) = \mathbf{C}^{n^2}$ and $f(x) = \det x$.

$$
\begin{array}{lll}
(0) & & 0 \\
| & s+1 & \\
(1) & & -s-1/2 \\
| & s+2 & \\
(2^2) & & -2s-4/2 \\
\vdots & & \\
| & s+n & \\
(n^2) & & -ns-n^2/2
\end{array}
\qquad b_f(s) = \prod_{j=1}^{n} (s+j).
$$

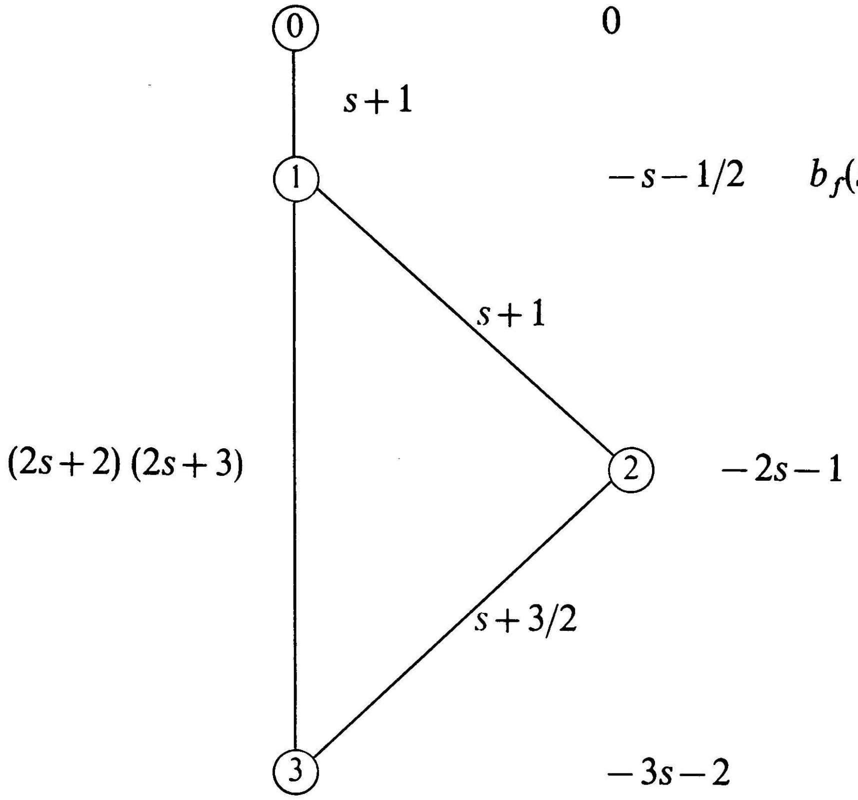
Here $a$ means a good Lagrangean which is the conormal bundle to an $a$-codimensional submanifold. $\bigcirc$—$\bigcirc$ means that the two corresponding good Lagrangeans have a good intersection.

The polynomial attached to the intersection is the ratio of the corresponding $b_\Lambda$-functions, calculated by Theorem 11.8.2. The polynomial attached to the circle is the order of $f^\lambda$.

(ii) $X = \mathbf{C}^n$, $f(x) = x_1^2 + \ldots + x_n^2$

$$
\begin{array}{lll}
(0) & & 0 \\
| & s+1 & \\
(1) & & -s-1/2 \\
| & s+n/2 & \\
(n) & & -2s-n/2
\end{array}
\qquad b(s) = (s+1)(s+n/2)
$$

(iii) $X = \mathbf{C}^3$ $f = x^2y + z^2$




$$b_f(s) = (s+1)^2(s+3/2)$$