§6. Existence of the two-variable polynomial

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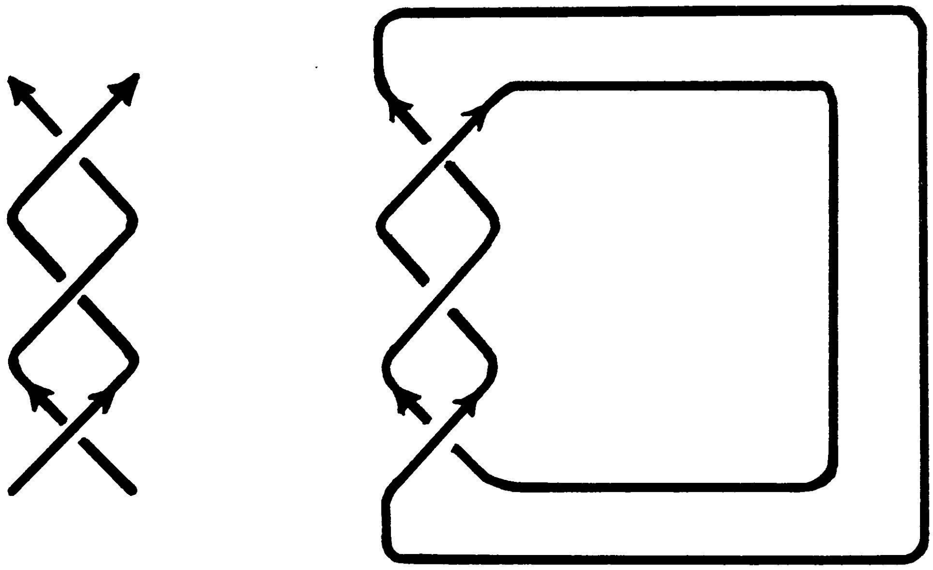
The polynomial will be defined as a braid invariant.

A braid α gives rise to a link $K(\alpha)$ by the "closing" operation, as shown in the picture



$\alpha \in B_2$ $K(\alpha)$

Closing a braid

Recall that every oriented link is ambient isotopic to a closed braid, as was already known to Alexander [Al]. (See also [Mo].)

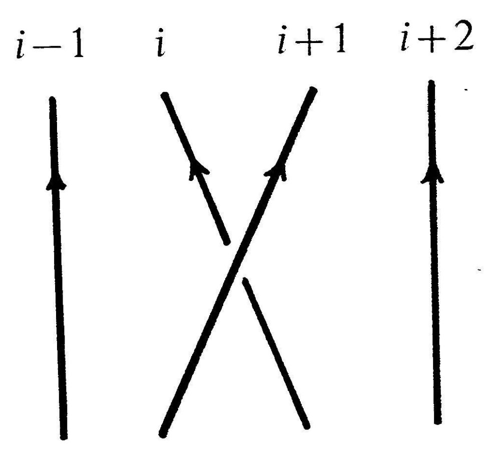
Now, let $K = \mathbf{C}(q, z)$ be the rational field in 2 variables q, z over the complex numbers and let $w = 1 - q + z \in K$.

With every braid $\alpha \in B_n$ we will associate an element $V_\alpha(q, z)$ in the quadratic extension $K(\sqrt{q/zw})$ of K.

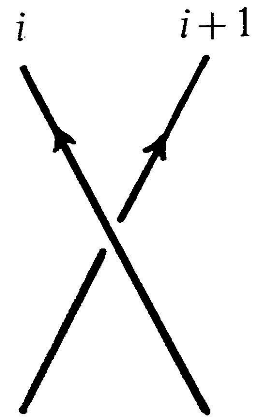
It is a quite remarkable fact that $V_\alpha(q, z)$ will depend only on the link $K(\alpha)$ obtained from α by closing the braid as shown above, and not on α itself.

Thus, we will be able to define $V_K(q, z) = V_\alpha(q, z)$, where α is any braid such that K is ambient isotopic to $K(\alpha)$.

In order to define $V_\alpha(q, z)$ we now proceed to fix some notations. We use the following conventions regarding the generators $\sigma_1, ..., \sigma_{n-1} \in B_n$ of the braid group on n strings B_n.



σ_i



σ_i^{-1}

Recall that B_n has a presentation on the generators $\sigma_1, \ldots, \sigma_{n-1}$ with relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{if} \quad |i-j| \geqslant 2,$$

and

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

for $i = 1, \ldots, n-2$.

Note that there is a well defined homomorphism $e: B_n \to \mathbf{Z}$ given by $e(\sigma_i) = 1, i = 1, \ldots, n-1$, on the generators. We call e the exponent sum.

There is also an obvious representation $\rho: B_n \to H_n$ determined by

$$\rho(\sigma_i) = T_i .$$

Note that $T_i \in H_n$ is invertible in H_n: $T_i^{-1} = \dfrac{1}{q}(1-q+T_i)$. Now, let $\alpha \in B_n$. The corresponding element $V_\alpha(q, z) \in K(\sqrt{q/zw})$ is defined by the formula

$$V_\alpha(q, z) = (1/z)^{(n+e(\alpha)-1)/2} \,.\, (q/w)^{(n-e(\alpha)-1)/2} \,.\, \mathrm{Tr}(\rho(\alpha)) ,$$

where $w = 1 - q + z$, $\rho: B_n \to H_n$ is as above, and $e(\alpha)$ is the exponent sum of α.

In order to show that $V_\alpha(q, z)$ depends only on the link $K(\alpha)$, we appeal to Markov's theorem which gives necessary and sufficient conditions for 2 braids $\alpha \in B_m$, $\beta \in B_n$ to produce isotopic links $K(\alpha)$, $K(\beta)$ by closing.

Define a Markov move of type 1 to be the operation of replacing a braid $\alpha \in B_n$ by a conjugate $\gamma\alpha\gamma^{-1} \in B_n$ with $\gamma \in B_n$.

A Markov move of type 2 consists in replacing $\alpha \in B_n$ by $\alpha . \sigma_n$ or $\alpha . \sigma_n^{-1}$ in B_{n+1}. Or, replacing $\alpha . \sigma_n \in B_{n+1}$, resp. $\alpha . \sigma_n^{-1} \in B_{n+1}$ by $\alpha \in B_n$ if α is a word in the generators $\sigma_1, \ldots, \sigma_{n-1}$ only.

THEOREM (Markov). *Let* $\alpha \in B_m$, $\beta \in B_n$ *be two braids. Then,* $K(\alpha)$ *and* $K(\beta)$ *are ambient isotopic as oriented links iff there exists a finite sequence of Markov moves carrying* α *to* β.

For a proof, see [Mo].

Thus, we have to show that $V_\alpha(q, z)$ is unchanged by Markov moves on α.

Let $\alpha \in B_n$, $\gamma \in B_n$ and $\beta = \gamma\alpha\gamma^{-1}$. Then the string numbers of α and β are the same. Also $e(\alpha) = e(\beta)$, and $\mathrm{Tr}(\rho(\beta)) = \mathrm{Tr}(\rho(\gamma)\rho(\alpha)\rho(\gamma)^{-1}) = \mathrm{Tr}(\rho(\alpha))$. Hence, $V_\beta(q, z) = V_\alpha(q, z)$.

If $\alpha \in B_n$ and $\beta = \alpha . \sigma_n \in B_{n+1}$, we have $e(\beta) = e(\alpha) + 1$, $n(\beta) = n + 1$ (where $n = n(\alpha)$). Thus,

$$\begin{aligned} V_\beta(q, z) &= (1/z)^{(n+e(\alpha)-1)/2} . (q/w)^{(n-e(\alpha)-1)/2} . (1/z) . \mathrm{Tr}(\rho(\alpha) . T_n) \\ &= V_\alpha(q, z), \text{ as desired}. \end{aligned}$$

If $\alpha \in B_n$ and $\beta = \alpha\sigma_n^{-1} \in B_{n+1}$, then $e(\beta) = e(\alpha) - 1$, $n(\beta) = n + 1$ and

$$V_\beta(q, z) = (1/z)^{(n+e(\alpha)-1)/2} . (q/w)^{(n-e(\alpha)-1)/2} . (q/w) . \mathrm{Tr}\{\rho(\alpha) . T_n^{-1}\} .$$

Now,

$$\rho(\alpha) . T_n^{-1} = \frac{1}{q} \rho(\alpha) . (1 - q + T_n) ,$$

and

$$\mathrm{Tr}\{\rho(\alpha) . T_n^{-1}\} = \frac{1}{q} (1 - q + z) . \mathrm{Tr}(\rho(\alpha)) = (w/q) . \mathrm{Tr}(\rho(\alpha)) .$$

Hence, again $V_\beta(q, z) = V_\alpha(q, z)$.

Thus $V_\alpha(q, z)$ is well defined as an invariant of oriented links.

On the face of it, $V_\alpha(q, z)$ does not look much so far like a polynomial with integral coefficients. However, as it turns out, a slick change of variables will do the trick. We have:

PROPOSITION 6.1. *There exists for each link* K, *a unique* Laurent *polynomial* $P_K(l, m) \in \mathbf{Z}[l, l^{-1}, m, m^{-1}]$, *such that*

$$P_K\big(i(z/w)^{1/2},\, i(q^{-1/2}-q^{1/2})\big) = V_\alpha(q, z)\,,$$

whenever α *is a braid giving rise (up to ambient isotopy, of course) to the link* K *by closing.*

Thus, $V_\alpha(q, z)$ becomes the Laurent polynomial $P_{K(\alpha)}(l, m)$ in

$$\mathbf{Z}[l, l^{-1}, m, m^{-1}]$$

after the change of variables

$$l = i(z/w)^{1/2}\,, \qquad m = i(q^{-1/2}-q^{1/2})\,.$$

The key to the proof of this fact is the skein invariance of $V_\alpha(q, z)$ which we now proceed to show.

Let $\beta, \gamma \in B_n$ be two braids and let $\alpha_+, \alpha_-, \alpha_0$ be the three braids

$$\alpha_+ = \beta\sigma_k\gamma\,, \qquad \alpha_- = \beta\sigma_k^{-1}\gamma\,, \qquad \alpha_0 = \beta\gamma\,,$$

for some index $k \leqslant n-1$.

For any braid $\alpha \in B_n$, with exponent sum $e = e(\alpha)$, define

$$W_\alpha(q, z) = (1/z)^{(n+e-1)/2} \,.\, (q/w)^{(n-e-1)/2} \,.\, \rho(\alpha) \in H_n\,,$$

where H_n is now the Hecke algebra over $K(\sqrt{q/zw})$ with $K = \mathbf{C}(q, z)$, corresponding to q.

Skein invariance lemma. *Set* $l = i(z/w)^{1/2}$ *and* $m = i(q^{-1/2}-q^{1/2})$. *Then, we have the relation*

$$lW_{\alpha_+} + l^{-1}W_{\alpha_-} + mW_{\alpha_0} = 0\,.$$

Taking the trace, we obtain from this lemma the skein invariance of $V_\alpha(q, z)$: With the same notations as above

$$lV_{\alpha_+} + l^{-1}V_{\alpha_-} + mV_{\alpha_0} = 0\,.$$

Proof of the lemma. Set $e = e(\alpha_0)$, and observe that we have

$$W_{\alpha_+} = (1/z)^{1/2}(q/w)^{-1/2}(1/z)^{(n+e-1)/2}(q/w)^{(n-e-1)/2} \,.\, \rho(\beta)T_k\rho(\gamma)\,,$$
$$W_{\alpha_-} = (1/z)^{-1/2}(q/w)^{1/2}(1/z)^{(n+e-1)/2}(q/w)^{(n-e-1)/2} \,.\, \rho(\beta)T_k^{-1}\rho(\gamma)\,.$$

An easy calculation now gives for $lW_{\alpha_+} + l^{-1}W_{\alpha_-} + mW_{\alpha_0}$ an expression of the form

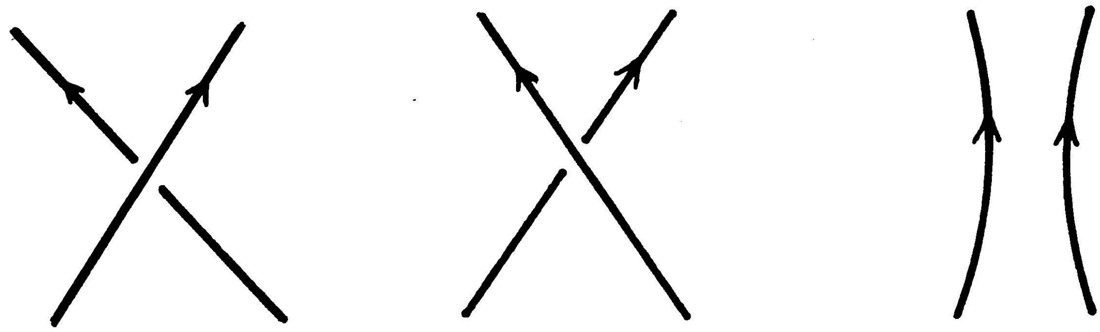
$$i(1/z)^{(n+e-1)/2}(q/w)^{(n-e-1)/2} \,.\, \rho(\beta) \,.\, C \,.\, \rho(\gamma)\,,$$

where

$$C = q^{-1/2}T_k - q^{1/2}T_k^{-1} + q^{-1/2} - q^{1/2}\,.$$

Recalling that $T_k^{-1} = q^{-1}(1-q+T_k)$, it is easy to verify that $C = 0$.

Proof of proposition 6.1. Let K_+, K_- and K_0 be three skein related links.



K_+ K_- K_0

It is an obvious consequence of the classical proof of Alexander's theorem that the three links can be presented as closed braids of the form

$$K_+ = K(\alpha_+)\,, \quad K_- = K(\alpha_-)\,, \quad K_0 = K(\alpha_0)\,,$$

with $\alpha_+ = \beta\sigma_k\gamma$, $\alpha_- = \beta\sigma_k^{-1}\gamma$, $\alpha_0 = \beta\gamma$ for some braids $\beta,\ \gamma \in B_n$ and some index $k \leqslant n-1$.

Writing $V(K)$ for V_α if $K = K(\alpha)$, it follows from the skein invariance lemma above, that

$$lV(K_+) + l^{-1}V(K_-) + mV(K_0) = 0\,,$$

if K_+, K_- and K_0 are skein related.

It follows now by induction on the link complexity, as in the proof of uniqueness in § 3, that $V(K)$ is actually a Laurent polynomial with integer coefficients in the variables l and m.

We change notation and set $P_K(l, m) \in \mathbf{Z}[l, l^{-1}, m, m^{-1}]$, where

$$P_{K(\alpha)}\big(i(z/w)^{1/2}, i(q^{-1/2}-q^{1/2})\big) = V_\alpha(q, z)$$

Since it is obvious that $P_K(l, m) = 1$ if K is the unknot $\bigcirc$, we have shown that $P\colon \mathscr{L} \to \mathbf{Z}[l, l^{-1}, m, m^{-1}]$ exists as a skein invariant. It is universal by what we saw before in § 3.