§7. Some properties of $P_K(l, m)$

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§ 7. Some properties of $P_K(l, m)$

In this paragraph we gather some of the basic properties of the polynomial $P_K(l, m)$, also denoted $P(K)$ if the variables are understood.

Let K' be the oriented link obtained from K by reversing the orientations of all the components. Then, we have

Property 7.1. $P(K') = P(K)$.

Proof. Let K_+, K_-, K_0 be three skein related links. We see that K'_+, K'_- and K'_0 are also skein related. Hence,

$$lP(K'_+) + l^{-1}P(K'_-) + mP(K'_0) = 0 .$$

By uniqueness, this implies $P(K') = P(K)$ for all K. (Of course $\bigcirc' = \bigcirc$.)

Property 7.1 can also be proved from the definition given in § 6 as follows. If $K = K(\alpha)$, then $K' = K(\alpha')$, where $\alpha' = \sigma_{i_r}^{\varepsilon_r} \dots \sigma_{i_1}^{\varepsilon_1}$ if $\alpha = \sigma_{i_1}^{\varepsilon_1} \dots \sigma_{i_r}^{\varepsilon_r}$. Observe that the operation $\alpha \mapsto \alpha'$ is a well defined antiautomorphism of B_n. There is an analogous antiautomorphism of H_n, sending the monomial $M = T_{i_1} \dots T_{i_r}$ to $M' = T_{i_r} \dots T_{i_1}$ and it is easily checked that for all $x \in H_n$, $\mathrm{Tr}(x) = \mathrm{Tr}(x')$.

Next, let $K^\times$ be the mirror image of K. Then we have

Property 7.2. $P_{K^\times}(l, m) = P_K(l^{-1}, m)$.

Proof. Observe that if K_+, K_-, and K_0 are skein related, then so are $K_-^\times$, $K_+^\times$ and $K_0^\times$ *in this order*, i.e.

$$lP(K_-^\times) + l^{-1}P(K_+^\times) + mP(K_0^\times) = 0 .$$

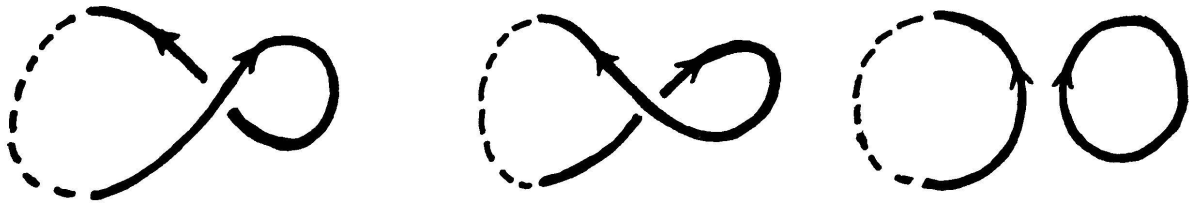
The property follows by uniqueness applied to $P_{K^\times}(l, m) = P_K(l^{-1}, m)$.

We shall skip the alternative proof of that property based on braid presentations.

If K_1 and K_2 are two links and $K_1 \amalg K_2$ their distant union (disjoint, unlinked), then we have

Property 7.3. $P(K_1 \amalg K_2) = -\dfrac{l + l^{-1}}{m} \,.\, P(K_1) \,.\, P(K_2)$.

Proof. If $K_2 = \bigcirc$, this follows from the skein invariance as shown in the following picture



$L_1 = L_+$ $\qquad$ $L_1 = L_-$ $\qquad$ $L_0 = L_1 \amalg \bigcirc$

which yields

$$lP(K_1) + l^{-1}P(K_1) + mP(K_1 \amalg \bigcirc) = 0\,,$$

and therefore

$$P(K_1 \amalg \bigcirc) = -\frac{l + l^{-1}}{m}\,.\,P(K_1)\,.$$

If K_2 is more complicated, use induction on the complexity of one of its diagrams L_2. If L_2^+, L_2^-, L_2^0 are skein related, so are $L_1 \amalg L_2^+$, $L_1 \amalg L_2^-$, $L_1 \amalg L_2^0$ for any diagram L_1 of K_1 and Property 7.3 follows.

Second proof. If $K_1 = K(\alpha)$ with $\alpha \in B_m$ and $K_2 = K(\beta)$, with $\beta \in B_n$, then $K_1 \amalg K_2 = K(\alpha\,.\,s(\beta))$ with $\alpha\,.\,s(\beta) \in B_{m+n}$, where $s\colon B_n \to B_{m+n}$ shifts all indices of the generators $\sigma_1, \ldots, \sigma_{n-1}$ by m, i.e. $s(\sigma_i) = \sigma_{m+i}$. It follows that α and $s(\beta)$ commute in B_{m+n}, and it is easily verified that $\mathrm{Tr}(\rho(\alpha\,.\,s(\beta))) = \mathrm{Tr}(\rho(\alpha))\,.\,\mathrm{Tr}(\rho(\beta))$. Then,

$$V_{\alpha s(\beta)}(q, z) = (q/zw)^{1/2}\,.\,V_\alpha(q, z)\,.\,V_\beta(q, z)\,.$$

With $l = i(z/w)^{1/2}$ and $m = i(q^{-1/2} - q^{1/2})$, we have

$$-\frac{l + l^{-1}}{m} = -\frac{(z/w)^{1/2} - (z/w)^{-1/2}}{q^{-1/2} - q^{1/2}} = -\left(\frac{zq}{w}\right)^{1/2} \frac{1 - \dfrac{w}{z}}{1 - q}$$

$$= -\left(\frac{zq}{w}\right)^{1/2} \frac{z - (1 - q + z)}{z(1-q)} = \left(\frac{q}{zw}\right)^{1/2}$$

Therefore,

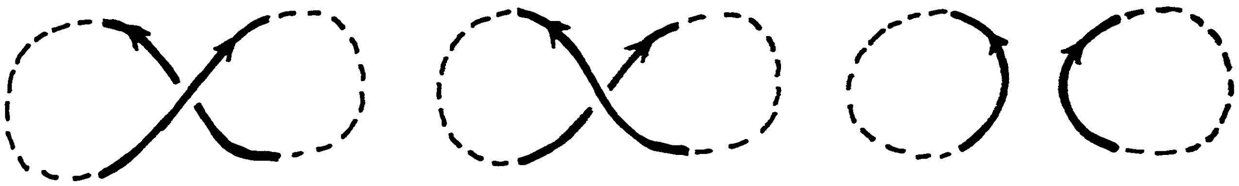
$$V_{\alpha s(\beta)}(q, z) = -\frac{l + l^{-1}}{m}\,.\,V_\alpha(q, z)\,.\,V_\beta(q, z)$$

as required.

If K_1, K_2 are 2 links, denote by $K_1 \# K_2$ a connected sum of K_1 and K_2 performed from the unlinked union on any choice of components.

PROPERTY 7.4. $P(K_1 \# K_2) = P(K_1) \,.\, P(K_2)$.

Proof. We use the skein relation



$L_+ = L_1 \# L_2$ $\qquad$ $L_- = L_1 \# L_2$ $\qquad$ $L_0 = L_1 \amalg L_2$

where L_1 and L_2 are diagrams of K_1 and K_2.

This gives the formula

$$lP(L_1 \# L_2) + l^{-1}P(L_1 \# L_2) + mP(\overline{L_1 \amalg L_2}) = 0\,.$$

Solving for $P(L_1 \# L_2)$ and using property 7.3, the factor $-(l+l^{-1})/m$ cancels out and the result follows.

The proof using braid presentations is more complicated and will be omitted.

Since $P: \mathscr{L} \to \mathbf{Z}[l, l^{-1}, m, m^{-1}]$ is the universal skein invariant, it must specialize to the Alexander polynomial and to the one-variable Jones polynomial.

Specifically, define

$$\Delta_K(t) = P_K\big(i, i(t^{1/2} - t^{-1/2})\big)\,,$$

then we have

PROPERTY 7.5. $\Delta_K(t)$ *satisfies the skein invariance*

(1) $\Delta_{\bigcirc}(t) = 1\,,$

(2) $\Delta(K_+) - \Delta(K_-) + (t^{1/2} - t^{-1/2})\Delta(K_0) = 0\,,$

which characterizes the Alexander *polynomial as normalized by* J. Conway. (See L. Kauffman, [Ka_1].)

Recall from § 3 that the exponent of m in each monomial of $P_K(l, m)$ is congruent mod 2 to $r(K) - 1$, where $r(K)$ is the number of components of K. Hence, for a knot, a link with a single component, the exponent of m in $P_K(l, m)$ is even and therefore $\Delta_K(t) = P_K(i, i(t^{1/2} - t^{-1/2}))$ is indeed a Laurent polynomial in t.

To obtain the one-variable Jones polynomial we use the substitution $l = it$, $m = i(t^{1/2} - t^{-1/2})$. Explicitly,

$$V_K(t) = P_K(it, i(t^{1/2} - t^{-1/2}))$$

Then we have

PROPERTY 7.6. *$V_K(t)$ satisfies the skein invariance*

$$tV(K_+) - t^{-1}V(K_-) + (t^{1/2} - t^{-1/2})V(K_0) = 0\,,$$

which (together with $V(\bigcirc) = 1$) characterizes Jones *one-variable polynomial, with the sign conventions used in reference* [Jo_3].

Whereas $P_K(l, m)$ determines $\Delta_K(t)$ and $V_K(t)$, it is known that there are no other relations between these polynomials. More precisely:

(1) The Alexander polynomial $\Delta_K(t)$ does not determine Jones polynomial $V_K(t)$ because the trivial knot $\bigcirc$ and Conway's eleven crossing knot 11_{471} have $\Delta(t) = 1$, but $V_K(t) \neq 1$ for $K = 11_{471}$.

(2) $V_K(t)$ does not determine $\Delta_K(t)$: The knots 4_1 and 11_{388} have the same $V(t)$ but different $\Delta(t)$.

(3) $V_K(t)$ and $\Delta_K(t)$ together do not determine $P_K(l, m)$: The knot 11_{388} and its mirror image have the same $V(t)$ and $\Delta(t)$ but different $P(l, m)$.

For more details on these questions, see [L.-M.].

We now turn to L. Kauffman's definition of the one-variable Jones polynomial $V_K(t)$ directly from the link diagram.

§ 8. L. KAUFFMAN'S APPROACH TO V. JONES' ONE-VARIABLE POLYNOMIAL

The importance of Kauffman's approach [Ka_3] is that it gives a new way to define and compute Jones polynomial $V_K(t)$. It is by using this definition that Kauffman and Murasugi prove their theorems about alternating links (see § 10 and 11).

Let L be an *unoriented* link diagram. Look at a double point; with no string orientation, they all look the same, up to a local homeomorphism: