§9. Tait conjectures

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Now, if one uses our computation in § 3

$$P(T_+) = -2a_-a_+^{-1} - a_-^2a_+^{-2} + a_+^{-2}a_0^2$$

and substitutes $a_+ = l, a_- = l^{-1}, a_0 = m$ one gets

$$P_{T_+}(l, m) = (-2l^{-2}-l^{-4})m^0 + l^{-2}m^2 .$$

The last substitution $l = it$; $m = i(t^{1/2}-t^{-1/2})$ gives (with relief!) the same result for Jones one variable polynomial. (Bulletin AMS definition.)

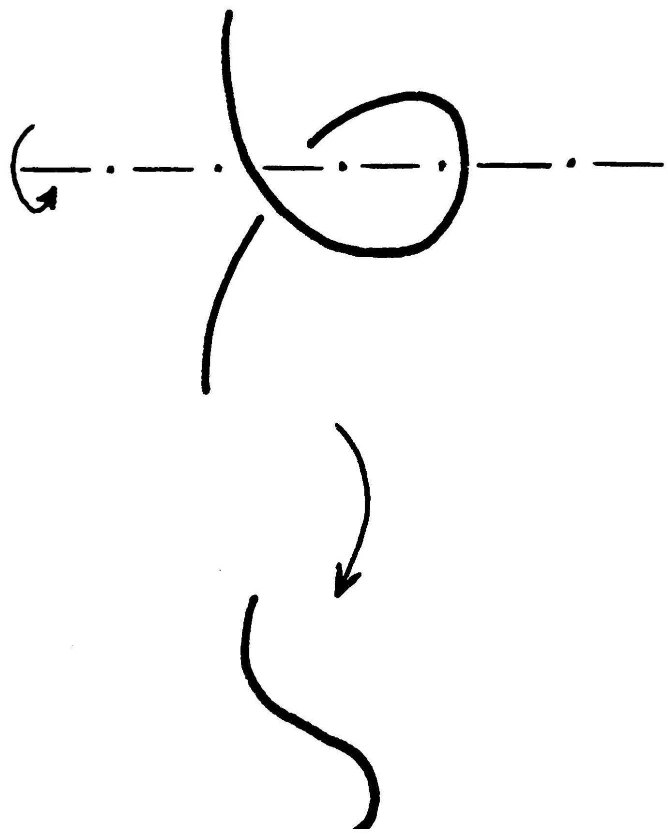
§ 9. Tait conjectures

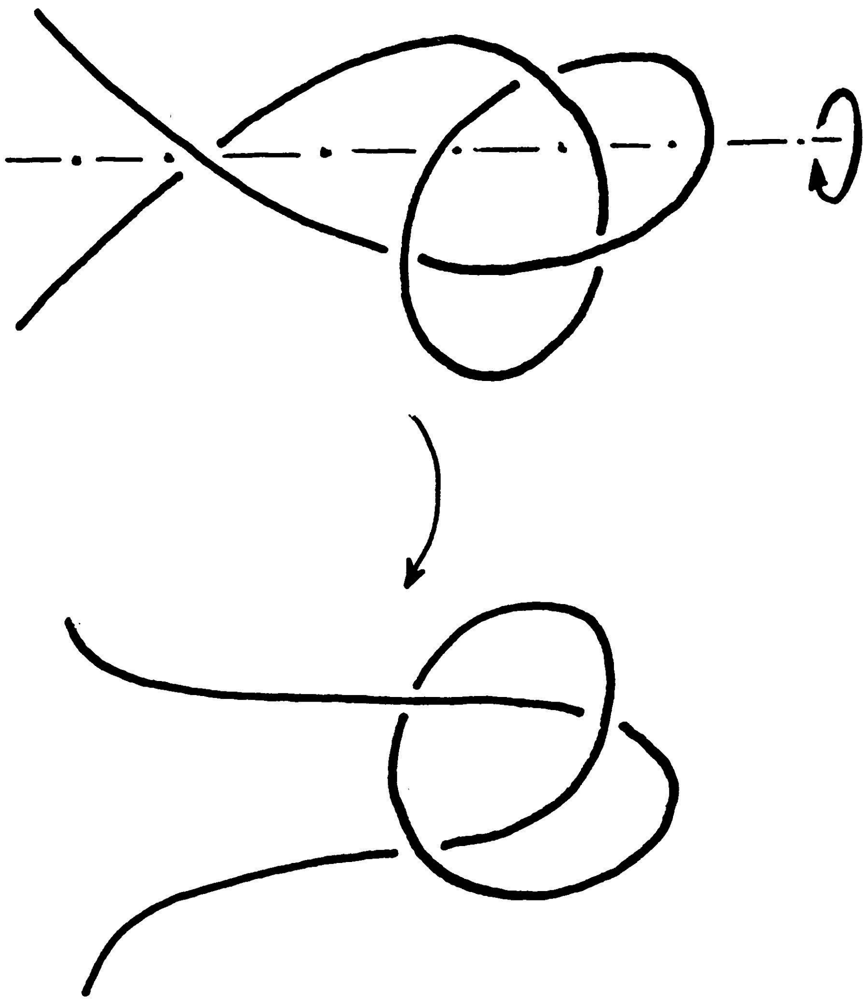
Tait was primarily interested in the classification of knots (i.e. one component links). He organized the job in two steps.

Step 1. Classify generic immersions of the circle in S^2 (not $\mathbf{R}^2$!) modulo homeomorphisms (possibly orientation reversing) of S^2. This was mostly done by the Rev. T. P. Kirkman (around 1880).

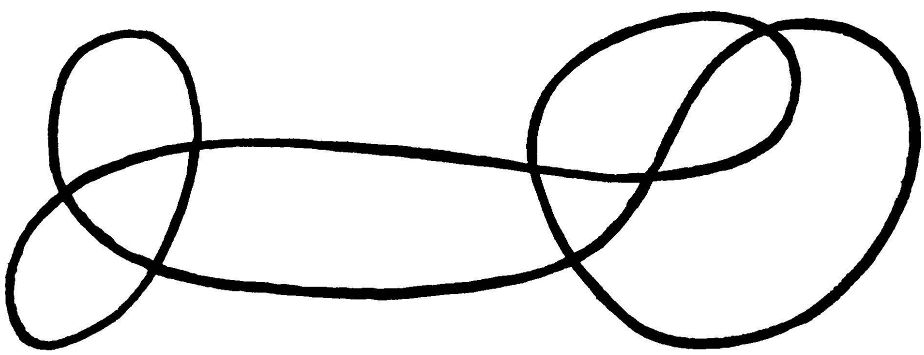
In this process, one has to remember that one is looking at knots in $\mathbf{R}^3$ and that one is trying to list knots according to their "knottiness", i.e. their minimal crossing number. So, Tait first reduced the number of double points of a generic immersion by making one "local 180° rotation".

Examples.





So, really, the problem was to list *reduced* generic immersion of S^1. Tait also recognised that is was sufficient to classify "prime" immersions, i.e. immersions indecomposables with respect to connected sum.
Example of a connected sum:



Step 2. Find how many knot types correspond to the same generic immersion. Tait's first observation was:

PROPOSITION 9.1. *A link projection being given, one can always choose the heights at the double points in order that the corresponding link diagram be alternating.*

By definition, a link diagram is alternating if, when one follows any string, the crossings are alternatively over and under.

We now reproduce Tait's proof, because it will play its part in § 11.

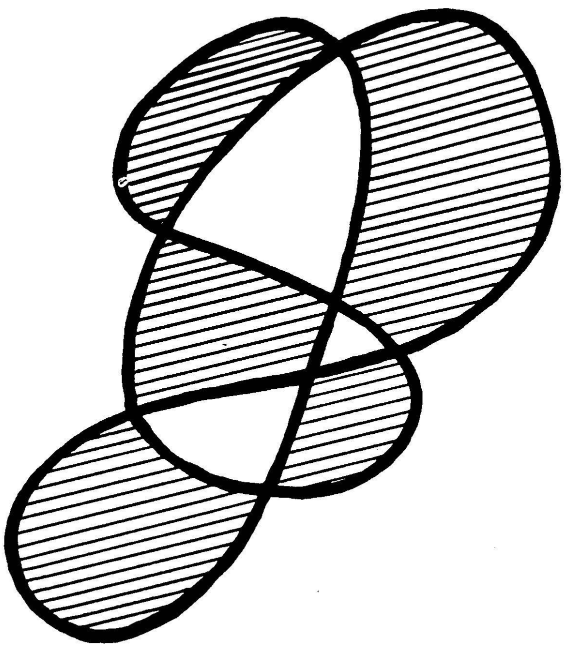
Proof of proposition 9.1. Let L be a link projection in S^2, not passing through the north pole N.

Call "region" a connected component of $S^2 - L$.

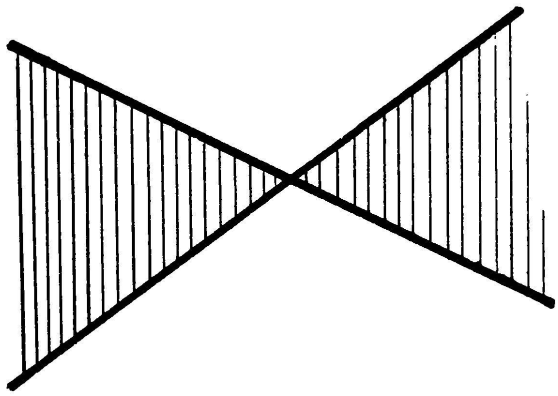
If $P \in S^2 - L$, let $I(P)$ be the intersection number mod 2 of L and a generic 1-chain joining P to N.

Shade the regions for which $I \equiv 1 \bmod 2$. S^2 is thus painted like a chessboard, the region containing N being unshaded.

Example.



Let X be a double point of L. Near X, two opposite regions are shaded and two aren't.



Choose a thread and travel along this thread toward the crossing point and a little further. Call this thread "*rl*" if the shaded region is first on your right and then on your left, while you travel. Notice that this does not depend on the orientation you choose on the thread.

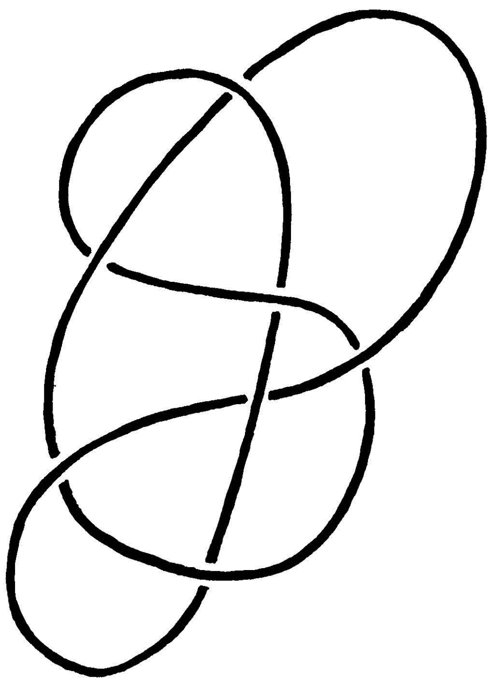
At each double point, one thread will be "*rl*" and the other will be "*lr*".

To construct an alternating link diagram from the link projection L we make the following *convention*: A "*rl*" thread always passes over a "*lr*" thread.

ASSERTION. *The link diagram thus obtained is alternating.*

Proof. If one follows a string, after a double point a "*rl*" thread becomes a "*lr*" thread and conversely. Q.E.D.

Picture:



Suppose that L is a connected link projection. There are exactly two ways to obtain an alternating link diagram from it. In this setting, the question of amphicheirality is very natural: Are the two links ambient isotopic? If yes, they are amphicheiral (nowadays, one also says achiral). If not, they are now called "chiral".

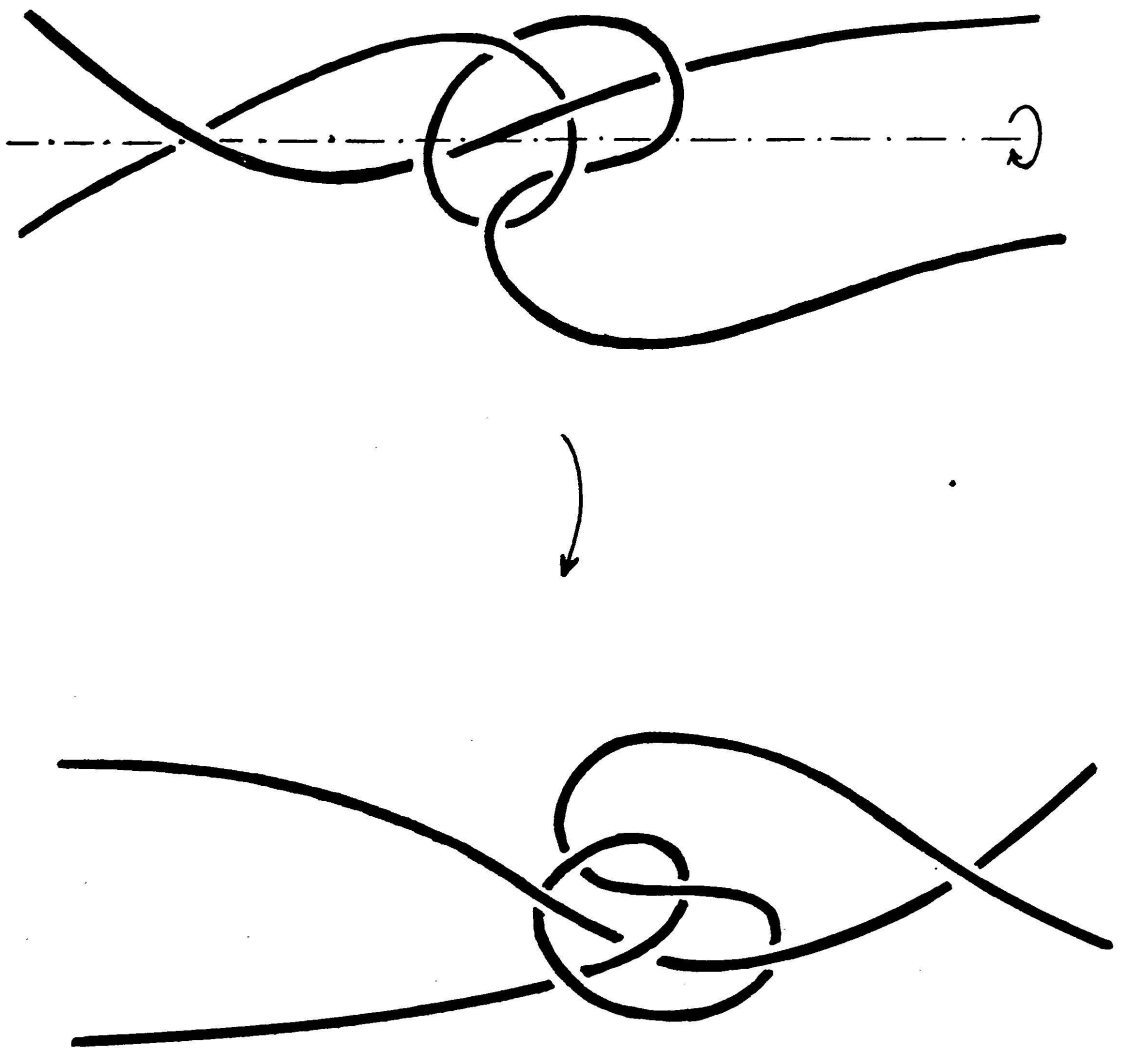
Roughly speaking the chirality question arose more or less in these terms in Tait. It is however obscured by considerations pertaining to knot projections rather than to knots in $\mathbf{R}^3$.

In order to classify alternating knots, Tait used the following principles, now called Tait conjectures:

Conjecture A. Two reduced alternating diagrams of the same alternating knot have the same number of crossing points. This number is minimal among all diagrams.

A stronger form of conjecture A would be: The minimal diagrams of an alternating knot are exactly the reduced alternating ones.

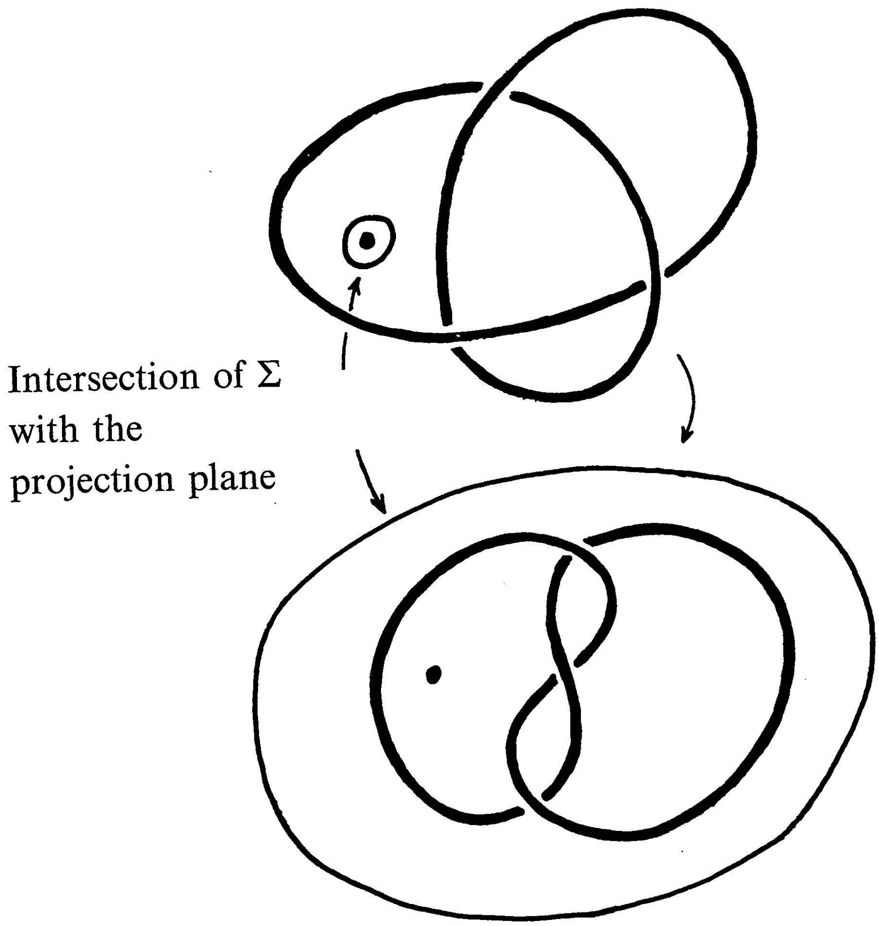
Conjecture B. Two reduced alternating diagrams of the same knot are "essentially unique". More precisely one can pass from one to another by a sequence of the following two operations:



(i) Another kind of "local 180° rotation" illustrated in the above picture, and called "twisting" by Tait. (An analogous operation is called by him "distortion".)

(ii) An inversion with respect to a 2-sphere Σ in S^3 intersecting the projection "plane" in a circle, followed by a mirror through the projection plane (in order that the composition be orientation preserving). For that, Tait introduced the name "flype", an old Scottish word meaning "to turn outside in".

Example.



Remarks. 1. If conjectures A and B were true, the classification of alternating knots would mainly rely on listing generic immersions of S^1 in S^2.

2. If conjecture A is true, then an alternating reduced knot diagram with at least one crossing point represents a non trivial knot. This was first proved by C. Bankwitz, with a mistake corrected by R. Crowell. See [Ba], and [Cr].

3. Tait noticed that, from eight crossings on, there exist non alternating knots. No actual proof was given. Tait had no "principles" to classify non alternating knots.

4. Conjecture B is still open.

Let us now come back to the notion of writhe number of a knot diagram L defined in § 8. Recall that, by definition, $w(L)$ is the sum of the signs of the crossing points.

A topological interpretation of $w(L)$ is the following: take a small tubular neighborhood of L and restrict the projection onto $\mathbf{R}^2$ to the boundary of this neighborhood. This restriction will have two curves of singularities: the "contour apparent". Choose one of them; it is a parallel of the knot. The linking coefficient of this parallel with the knot is precisely $w(L)$. Notice that this parallel is defined only when a projection is chosen.

A careful reader of Tait [Tai] on p. 308 will remark that Tait knew that. The Gaussian integral, interpreted via Maxwell theory, takes place of the linking coefficient. In Tait's point of view the parallel is turned 90° downward on each fiber of the regular neighborhood of the knot.

C. N. Little also introduced the number $w(L)$. He used it to classify knots by making the following statement:

Little principle: Any two minimal diagrams of the same knot have the same writhe number. (See [Li].)

This principle is known to be false; a counter-example is given by Little's duplication: the knot diagrams listed in Rolfsen's book as 10_{161} and 10_{162} have distinct writhe number, but represent the same knot as discovered by K. Perko [Pe].

However, the following is still open:

CONJECTURE C. Any two reduced and alternating diagrams of an (alternating) knot have the same writhe number.

If L is a knot diagram, let $L^{\times}$ denote the mirror image of L. Clearly: $w(L) = -w(L^{\times})$. So, if one believes some of the above conjectures, one is ready to make the following conjecture, used by Tait as a fact:

CONJECTURE D. If K is an alternating and amphicheiral knot, then any minimal projection of K has an even number of double points.

More daring people would conjecture that minimal diagrams of an amphicheiral knot have Tait number zero (i.e. writhe number zero).

Helped by these statements, Tait gave a list of twenty knots up to ten crossings which are amphicheiral and believed that the list was complete (which it is!).

We conclude this paragraph by recalling a few dates:

a. First proof that knots do exist: H. Tietze in 1908 [Ti] proved that the trefoil is knotted.

b. First proof that non amphicheiral knots do exist: M. Dehn in 1914 [De] proved that the left handed trefoil is not ambient isotopic to the right handed trefoil.

c. First proof that non alternating knots do exist: R. Crowell [Cr] and K. Murasugi [Mu_1] proved in 1957 that the (3, 4) torus knot is non alternating. This result was already stated by C. Bankwitz.

§ 10. L. Kauffman's and K. Murasugi's results

Definition. Let $g(t) \in \mathbf{Z}[t^{\pm 1/2}]$ be a non-zero element:

$$g(t) = \sum_{i=n}^{m} a_i t^i, \quad i \in \frac{1}{2}\mathbf{Z}, \quad a_n \neq 0, \quad a_m \neq 0.$$

Define span $g(t) = m - n$.

In principle span $g(t) \in \frac{1}{2}\mathbf{Z}$. But, if $g(t)$ is the one variable Jones polynomial of an oriented link in S^3, the span of $g(t)$ will actually be an integer. To see that, use induction on complexity, like in § 3.

Definition. Let K be a link in S^3.

K is said to be *splittable* if there exists a 2-sphere $\Sigma \subset S^3$ such that:

1. $\Sigma \cap K = \emptyset$.
2. There is at least one component of K in each connected component of $S^3 - \Sigma$.

THEOREM 10.1. *Let $K \subset S^3$ be an oriented unsplittable link. Then:*

$$\text{span } V_K(t) \leqslant c(K).$$

Comments. (i) One can define the number $s(K)$ of split components of K. Then, theorem 10.1 generalizes to: