2. Symmetries of the Hopf fibrations with fibre S^1

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Hence the fibres of H which lie on the torus T_α form a circle of radius $\sin\alpha\cos\alpha$. But a circle of latitude on $S^2(1/2)$, located at distance α from the north pole, has radius $(1/2)\sin 2\alpha = \sin\alpha\cos\alpha$. It follows that there is a correspondence between fibres of H and points of $S^2(1/2)$ which is a Riemannian isometry, proving the proposition. QED

Besides being parallel, the fibres of the Hopf fibration are assembled in a very regular way. The following two geometric features give an expression of this regularity, and were important in [GWZ].

1) *Constancy Feature.* Refer again to the figure showing the Hopf fibration of S^3, in which we see S^3 decomposed into a pair of orthogonal great circles and a family of intermediating tori:

$$\begin{aligned} T_0 &= S^1(1) \times 0 \\ T_\alpha &= S^1(\cos\alpha) \times S^1(\sin\alpha) \qquad 0 < \alpha < \pi/2 \\ T_{\pi/2} &= 0 \times S^1(1)\,. \end{aligned}$$

Any two of these intermediating tori are a constant distance apart, and hence parallel to one another. There is a natural "radial projection" map between them, which matches closest neighbors on the two surfaces. It is easy to see that this map also matches Hopf circles, and in this sense we regard the Hopf fibration as "constant" on the family of tori. A corresponding phenomenon can be observed in all the Hopf fibrations.

2) *Inductive Feature.* A Hopf fibration contains within itself copies of lower dimensional Hopf fibrations, and can be regarded as assembled from these in a certain way. For example, just as C^n contains C^{n-1}, so does the Hopf fibration of S^{2n-1} contain the Hopf fibration of S^{2n-3}.

2. Symmetries of the Hopf fibrations with fibre S^1

Let $H: S^1 \hookrightarrow S^{2n-1} \to CP^{n-1}$ denote a Hopf fibration with fibre S^1. By a *symmetry* of H we mean a rigid motion of S^{2n-1} which takes Hopf circles to Hopf circles. We want to find these symmetries explicitly.

The unitary group

$$\begin{aligned} U(n) &= Gl(n, C) \cap O(2n) \\ &= \text{complex general linear group} \cap \text{orthogonal group} \end{aligned}$$

consists of complex linear maps which are also rigid. Since these maps take complex lines to complex lines, they must be symmetries of the above Hopf fibration.

But there are other symmetries. Define *complex conjugation*

$$c: C^n \to C^n \quad \text{by} \quad c(z_1, ..., z_n) = (\overline{z_1}, ..., \overline{z_n}).$$

Note that c lies in $O(2n)$ but not in $Gl(n, C)$, yet takes complex lines to complex lines, hence must be a symmetry of the Hopf fibration. Note also that c reverses the natural orientations of the complex lines in C^n.

The next proposition indicates that there are no further symmetries.

PROPOSITION 2.1. *The group G of all symmetries of the Hopf fibration H is $G = U(n) \cup c\, U(n)$.*

Let g be a rigid motion of S^{2n-1} taking complex lines to complex lines. In case g reverses the natural orientations of complex lines, compose it with c so as to preserve these orientations. The new g commutes with multiplication by i, hence is complex linear. Since it is also rigid, it lies in $U(n)$. QED

Remark. Note that all the symmetries are orientation preserving when n is even, while half are orientation reversing when n is odd.

The group of symmetries of the Hopf fibration is quite large, and this may be underscored by exhibiting symmetries with preassigned features. We collect some of these in the following proposition.

PROPOSITION 2.2. *Let $H: S^1 \hookrightarrow S^{2n-1} \to CP^{n-1}$ be a Hopf fibration. Then*

a) *There is a symmetry of H inducing the identity on the base space (and thus taking each Hopf circle to itself) and restricting to a preassigned rotation on a given Hopf circle.*

b) *If P and Q are any two fibres of the Hopf fibration, then any preassigned rigid motion of P onto Q can be extended to a symmetry of H.*

c) *The group of symmetries acts transitively on S^{2n-1}, and in particular acts transitively on fibres.*

By contrast, here is a limitation on the possible symmetries.

d) *There is no symmetry of* H *inducing the identity on the base space and reversing the orientations of the Hopf circles.*

Consider the symmetries $z \cdot Id$, $|z| = 1$, which multiply each coordinate in C^n by the complex number z of unit length. They induce the identity on the base space, and can be selected to take a fibre to itself by a preassigned rotation, proving a).

The transformations in $U(n)$ can take any complex line in C^n to any other by a preassigned orientation preserving rigid motion. Complex conjugation then adds the orientation reversing ones, proving b).

In particular, this implies c).

Suppose there were a symmetry of $H: S^1 \hookrightarrow S^{2n-1} \to CP^{n-1}$ taking each Hopf circle to itself with reversal of orientation. Then, by restriction to C^2, such a symmetry would also exist for $n = 2$. Its reversal of orientation on the total space S^3 would then contradict the remark following Proposition 2.1. This proves d). QED

Remarks. 1) Note that the existence of symmetries of H rotating each Hopf circle within itself shows again that these circles must be parallel.

2) Also note that a symmetry of $H: S^1 \hookrightarrow S^{2n-1} \to CP^{n-1}$ induces an isometry of the base space CP^{n-1} in its canonical metric. We remark without proof that *all* isometries of CP^{n-1} can be produced this way.

3. Hopf fibrations with fibre S^3

Choose orthonormal coordinates in R^{4n} and identify this space with quaternionic n-space H^n. A little care is needed in dealing with H^n because the quaternions form a *non*-commutative division algebra:

1) Scalars $v \in H$ will act on vectors $(u_1, ..., u_n) \in H^n$ from the *right.*

$$(u_1, ..., u_n)\, v = (u_1\, v, ..., u_n\, v)\,.$$

2) H-linear transformations of H^n will be expressed by matrices of quaternions acting from the *left* (so as to commute with scalar multiplication).

The quaternionic lines in H^n, each looking like a real 4-plane, form the quaternionic projective space HP^{n-1} and fill out H^n, with any two meeting only at the origin. The unit 3-spheres on these quaternionic lines give us the *Hopf fibration*

$$H: S^3 \hookrightarrow S^{4n-1} \to HP^{n-1}\,.$$