7. Symmetries of the Hopf fibration $H: S^7 \hookrightarrow S^{15} \rightarrow S^8$

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same is true if one of these planes is L_∞. Thus the Hopf 7-spheres on S^{15} are parallel to one another, as claimed. QED

The Riemannian metric on the base space S^8 which makes the Hopf projection $S^{15} \to S^8$ into a Riemannian submersion is that of a round 8-sphere of radius 1/2, which one sees directly just as in the previous cases.

7. Symmetries of the Hopf fibration $H: S^7 \hookrightarrow S^{15} \to S^8$

Proposition 7.1. *The group G of all symmetries of the Hopf fibration $H: S^7 \hookrightarrow S^{15} \to S^8$ is isomorphic to $Spin(9)$, the simply connected double cover of $SO(9)$.*

The action is as follows:

1) *There is a $g \in G$ inducing any preassigned orientation preserving isometry of the round base S^8, but no orientation reversing ones.*
2) *Given such a g, there is exactly one other symmetry,*

$$-g = \text{antipodal map} \circ g\,,$$

which induces the same action on S^8.

It is likely that Élie Cartan was aware of this result, since in [Ca 2, esp. pp. 424 and 466] he identified $Spin(9)$ as the group of isometries fixing a point in the Cayley projective plane CaP^2. It is not hard to see that this is the same as the group of symmetries of our Hopf fibration. The symmetry groups of the other Hopf fibrations can likewise be identified with the groups of isometries fixing a point in complex and quaternionic projective spaces, also known to Cartan.

We give the proof of Proposition 7.1 in a series of lemmas.

Lemma 7.2. *The only symmetries which take each fibre to itself are the identity and the antipodal map.*

Suppose $B: R^{16} \to R^{16}$ is such a symmetry. Since B maps

$$L_0 = \{(u, 0)\}\,,\ L_\infty = \{(0, v)\} \quad \text{and} \quad L_1 = \{(u, u)\}$$

into themselves, we must have

$$B(u, v) = \big(A(u), A(v)\big)$$

for some $A \in O(8)$. Since B maps $L_m = \{(u, mu)\}$ into itself, we get

$$B(u, mu) = \big(A(u), A(mu)\big) = \big(A(u), mA(u)\big).$$

Thus $$A(mu) = mA(u), \quad \text{all} \quad m, u \in Ca.$$

Now in this equation put $u = 1$ and keep m arbitrary:

$$A(m) = mA(1) = ma,$$

where we define $a = A(1)$. Insert this back into the previous equation, getting

$$(mu)a = m(ua), \quad \text{for all} \quad m, u \in Ca.$$

But then it follows from the nonassociativity of the Cayley numbers that the element a must be real. Since $A \in O(8)$, $a = \pm 1$. Thus $A(m) = \pm m$, and hence $B(u, v) = (\pm u, \pm v)$, that is, B is either the identity or the antipodal map, as claimed. QED

If we compare Lemma 7.2 with the corresponding assertions about the earlier Hopf fibrations, we conclude that the current Hopf fibration is the least symmetric of all.

LEMMA 7.3. *There is a symmetry of our Hopf fibration inducing any preassigned orientation preserving isometry of the base which keeps* L_0 *fixed.*

Such a symmetry must also take the orthogonal fibre $L_\infty = \{(0, v)\}$ to itself, and hence must be of the form

$$(u, v) \mapsto \big(A(u), B(v)\big), \quad \text{where} \quad A, B \in O(8).$$

Given such a symmetry, the Cayley line $L_m = \{(u, mu)\}$ is taken to the set $\{\big(A(u), B(mu)\big)\}$, which must itself be some Cayley line, say $L_{m'}$. Thus $B(mu) = m'\,A(u)$. Note that as a function of u, the left hand side is conformal with conformal factor $|\,m\,|$, while the right hand side is conformal with factor $|\,m'\,|$. Hence $|\,m\,| = |\,m'\,|$. Since the correspondence $m \mapsto m'$ is easily seen to be R-linear, it must be an isometry. Hence we can write $m' = C(m)$, with $C \in O(8)$.

Summarizing so far, a symmetry of our Hopf fibration which takes the fibre L_0 to itself must be of the form (A, B) with $A, B \in O(8)$, *and* there must exist a $C \in O(8)$ such that

$$B(mu) = C(m)\,A(u), \quad \text{for all} \quad m, u \in Ca.$$

Vice versa, if such a C exists, then the map (A, B) is indeed a symmetry of the Hopf fibration.

Since it is C which describes the induced action on the base space S^8, we need to be able to preassign $C \in SO(8)$. The possibility of doing this is the content of the "Triality Principle", as follows.

LEMMA 7.4. (Triality Principle for $SO(8)$, see [Ca 1, pp. 370 and 373] and [Fr]). *Consider the triples A, B and C in $SO(8)$ such that*

$$B(mu) = C(m)\,A(u)\,, \quad \textit{for all} \quad m, u \in Ca\,.$$

If any one of these three isometries is preassigned, then the other two exist and are unique up to changing sign for both of them.

We concentrate on preassigning C. Let G be the subset of $SO(8)$ consisting of all transformations C for which there exist A and B in $SO(8)$ satisfying the above equation for all $m, u \in Ca$. First note that G is actually a subgroup of $SO(8)$. For suppose that C and C' are in G, and correspond as above to A, B and A', B' respectively. Then

$$BB'(mu) = B\big(C'(m)\,A'(u)\big) = CC'(m)\,AA'(u)\,,$$

showing that $CC' \in G$. And similarly for inverses.

We want to show that G is all of $SO(8)$. Let x be an imaginary Cayley number of unit length. We claim

(7.5) *The right and left translations R_x and L_x are in G.*

To show this, we use the first two Moufang identities.

To satisfy $B(mu) = C(m)\,A(u)$ with $C = R_x$, choose $A = -L_x R_x$ and $B = R_x$. We must show that

$$(mu)x = -(mx)\,(xux)\,.$$

To do this, simply take the Moufang identity

$$z(xyx) = \big((zx)y\big)x$$

and put $x = x$, $y = u$ and $z = mx$, getting

$$(mx)\,(xux) = \big((mxx)u\big)x = -(mu)x\,,$$

since $x^2 = -1$. Thus $R_x \in G$.

To satisfy $B(mu) = C(m)\,A(u)$ with $C = L_x R_x$, choose $A = L_x$ and $B = -L_x$. We must show that

$$-x(mu) = (xmx)\,(xu)\,.$$

To do this, take the Moufang identity

$$(xyx)z = x(y(xz))$$

and put $x = x$, $y = m$ and $z = xu$, getting

$$(xmx)(xu) = x(m(xxu)) = -x(mu),$$

since $x^2 = -1$ as before. Thus $L_x R_x \in G$. Since we already know that G is a group and that it contains R_x, it must also contain L_x, establishing our claim.

Next we claim

(7.6) *The transformations* R_x *and* L_x, *as* x *ranges over all imaginary unit Cayley numbers, generate* $SO(8)$.

Since the subgroup G contains these transformations, this will show that G is all of $SO(8)$.

First note that any unit vector can be mapped to any other unit vector by a composition of such transformations. To see this, first suppose that u and v are orthogonal unit vectors: $\langle u, v\rangle = 0$. Then $\langle 1, vu^{-1}\rangle = 0$. Hence $x = vu^{-1}$ is an imaginary unit Cayley number such that $L_x(u) = (vu^{-1})u = v$. If u and v are unit vectors, but not necessarily orthogonal, just pick a unit vector w orthogonal to both. Find L_x and $L_{x'}$ such that $L_x(u) = w$ and $L_{x'}(w) = v$. Then $L_{x'}L_x(u) = v$, as desired.

So now it will be sufficient to show that any transformation in $SO(8)$ keeping 1 fixed is a composition of right and left translations by imaginary unit Cayley numbers. One such transformation is $-L_x R_x$ for any imaginary unit Cayley number x. Note that $-L_x R_x(x) = x$, so that this transformation also keeps x fixed. On the other hand, if y is an imaginary Cayley number orthogonal to x, then

$$-L_x R_x(y) = -xyx = xxy = -y,$$

since orthogonal imaginaries anti-commute by Fact 6. Thus $-L_x R_x$ is the identity on the 2-plane spanned by 1 and x, and is minus the identity on the orthogonal 6-plane. Viewed just on the imaginary Cayley numbers, this transformation is reflection about the line through x.

But it is easy to see that the set of reflections through all lines in R^7 generates $SO(7)$. Hence the transformations R_x and L_x, as x ranges over all imaginary unit Cayley numbers, generate $SO(8)$, as claimed.

Thus the subgroup G of transformations C in $SO(8)$, for which one can find A and B in $SO(8)$ satisfying $B(mu) = C(m)A(u)$ for all Cayley numbers m and u, must be all of $SO(8)$. In a similar fashion, one can preassign either A or B and find the other two, completing the proof of existence for the Triality Principle.

To prove uniqueness up to sign change for the Triality Principle, suppose C is the identity. Thus $B(mu) = m\,A(u)$ for all $m, u \in Ca$. Put $m = 1$ to learn that $B(u) = A(u)$. So now $A(mu) = m\,A(u)$. Put $u = 1$ to get $A(m) = m\,A(1) = ma$, where we define $a = A(1)$. Then put this back in the previous equation to get $(mu)a = m(ua)$. Since this holds for all $m, u \in Ca$, the element a must be real. Since A is orthogonal, $a = \pm 1$. Thus $A = B = \pm I$, proving uniqueness up to sign change when $C = I$. Uniqueness up to sign change for all $C \in SO(8)$ follows by composition. A similar argument gives uniqueness up to sign change when A or B is preassigned, completing the proof of the Triality Principle. QED

Preassigning C and using the Triality Principle to select A and B then completes the proof of Lemma 7.3: there is a symmetry of our Hopf fibration inducing any preassigned orientation preserving isometry of the base which keeps L_0 fixed. QED

We next use Lemma 7.3 to sharpen itself.

LEMMA 7.7. *There is a symmetry of our Hopf fibration inducing any preassigned orientation preserving isometry of the base. In particular, there is a symmetry taking any fibre to any other.*

On the base space S^8, we take the north pole to be L_0 and the south pole to be L_∞. Then the equator will consist of all L_m for which $|m| = 1$. Now consider the circle consisting of the points L_m for real m. We plan to show that this circle is contained in the orbit of L_0 under the symmetry group of H. Since this circle meets the equator in two points, L_1 and L_{-1}, we can then use (7.3) to conclude that the orbit of L_0 is all of S^8. Combining again with (7.3) will yield (7.7).

Consider the map $A_\phi : Ca^2 \to Ca^2$ defined by

$$A_\phi(u, v) = (\cos\phi\; u - \sin\phi\; v, \sin\phi\; u + \cos\phi\; v)\,.$$

These maps, for $0 \leqslant \phi \leqslant 2\pi$, provide a circle group of isometries of Ca^2. We claim

(7.8) *Each* A_ϕ *is a symmetry of our Hopf fibration.*

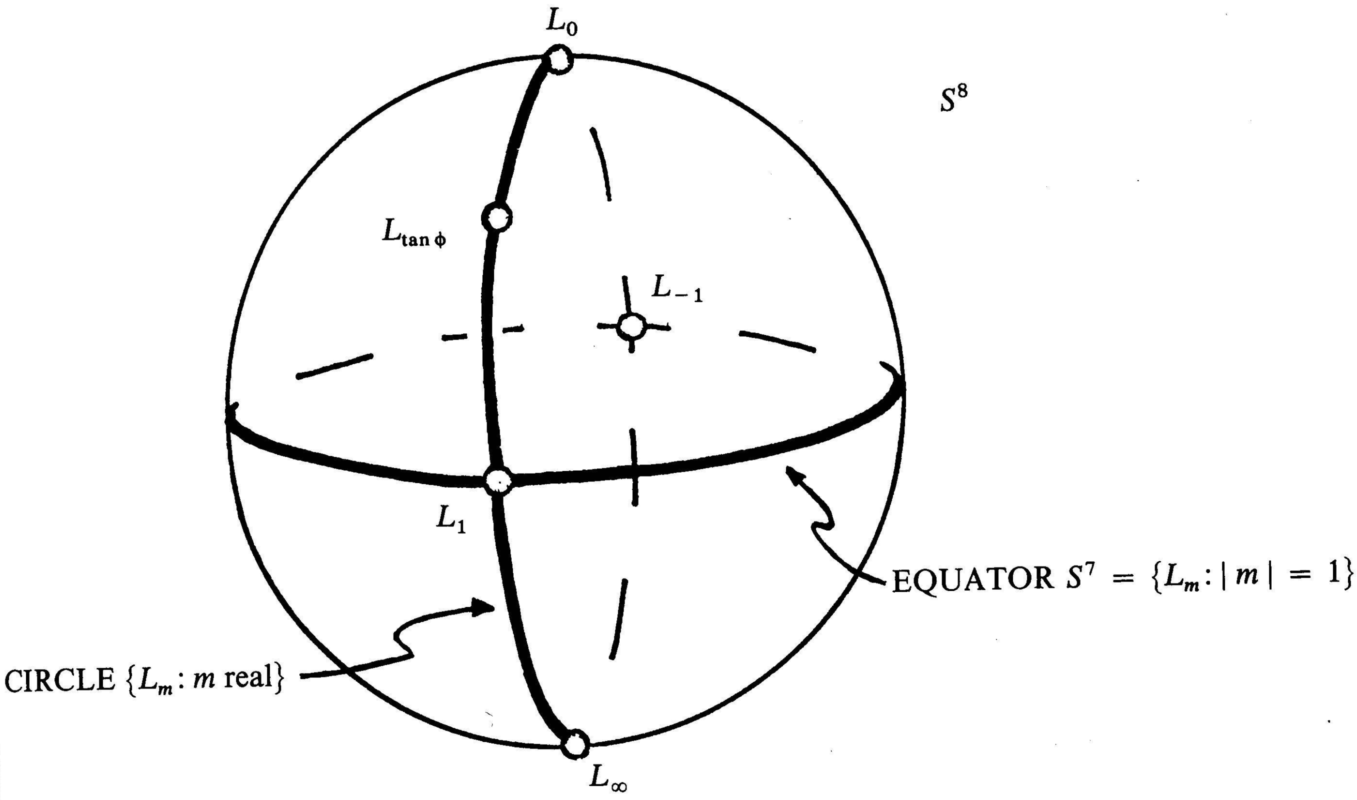

FIGURE 4

We must show that, given $m \in Ca$, there exists $m' \in Ca$ such that $A_\phi(L_m) = L_{m'}$. Now

$$\begin{aligned} A_\phi(u, mu) &= (\cos\phi\, u - \sin\phi\, mu, \sin\phi\, u + \cos\phi\, mu) \\ &= ((\cos\phi - \sin\phi\, m)\, u, (\sin\phi + \cos\phi\, m)\, u)\,. \end{aligned}$$

$$\begin{aligned} \text{Let} \qquad u' &= (\cos\phi - \sin\phi\, m)\, u \\ \text{and} \qquad m' &= (\sin\phi + \cos\phi\, m)(\cos\phi - \sin\phi\, m)^{-1}\,. \\ \text{Then} \qquad m'u' &= [(\sin\phi + \cos\phi\, m)(\cos\phi - \sin\phi\, m)^{-1}]\,[(\cos\phi - \sin\phi\, m)\, u]\,. \end{aligned}$$

The product on the right hand side may be reassociated because all the elements lie in the subalgebra of Ca spanned by the *two* elements m and u. As noted in section 5, such a subalgebra must be associative. But then clearly

$$m'u' = (\sin\phi + \cos\phi\, m)\, u\,,$$

so that we have

$$A_\phi(u, mu) = (u', m'u')\,.$$

Thus $A_\phi(L_m) = L_{m'}$, so each A_ϕ is a symmetry of our Hopf fibration, as claimed.

Since $A_\phi(L_0) = L_{\tan\phi}$, we see that the orbit of L_0 under the various A_ϕ is the circle $(L_m : m \text{ real})$. As indicated above, this is enough to complete the proof of (7.7). QED

LEMMA 7.9. *No symmetry of our Hopf fibration can induce an orientation reversing isometry of the base.*

Suppose there were such a symmetry. Using Lemma 7.7, we can assume it takes the fibre $L_0 = \{(u, 0)\}$ to itself. Then it must be of the form $(u, v) \mapsto (A(u), B(v))$ with $A, B \in O(8)$, and as we saw in (7.3) there must exist a $C \in O(8)$ such that $B(mu) = C(m)\, A(u)$ for all $m, u \in Ca$.

Composing our symmetry with an appropriate one guaranteed by Lemma 7.3 we can assume that $C(m) = \bar{m}$. Thus $B(mu) = \bar{m}\, A(u)$. Put $m = 1$ to conclude that $A = B$. Thus $A(mu) = \bar{m}\, A(u)$. Put $u = 1$ to conclude that $A(m) = \bar{m}\, A(1) = \bar{m}\, a$. Then put this back in the previous equation to get $(\overline{mu})a = \bar{m}(\bar{u}a)$. But $\overline{mu} = \bar{u}\,\bar{m}$ by Fact 4 of section 5. Hence

$$(\bar{u}\,\bar{m})a = \bar{m}(\bar{u}\,a)\,.$$

Now replace $\bar{u}$ by u and $\bar{m}$ by m to get

$$(um)a = m(ua) \qquad \text{for all} \qquad u, m \in Ca\,.$$

But this equation is impossible, which we see as follows.

Simply choose an automorphism of the Cayley numbers, see (5.4), which moves the element a to a unit quaternion. Apply such an automorphism to the above equation, and now consider that equation only for the quaternions:

$$(um)a = m(ua) \qquad \text{for all} \qquad u, m \in H\,.$$

But the quaternions are associative, so we remove the parentheses, then cancel the a and learn that

$$um = mu \qquad \text{for all} \qquad u, m \in H\,,$$

which is of course false. QED

Proof of (7.1). Let G again denote the group of all symmetries of the Hopf fibration $H: S^7 \hookrightarrow S^{15} \to S^8$. Consider the homomorphism $G \to O(9)$, which takes each $g \in G$ to its induced action on the base space S^8. By Lemma 7.9, the image lies in $SO(9)$. By Lemma 7.7, the homomorphism is onto. By Lemma 7.2, it is two-to-one. Thus G is a double covering of $SO(9)$. It remains to show that this covering is nontrivial.

It will be sufficient to look only at the symmetries of H which take the fibre $L_0 = \{(u, 0)\}$ to itself, and hence are of the form $(u, v) \mapsto (A(u), B(v))$. We already know that there must be a $C \in SO(8)$ such that $B(mu) = C(m)\, A(u)$ for all $m, u \in Ca$. To show that G is a nontrivial double covering of $SO(9)$, we must find a *loop* of C's which lifts to a *non-loop* of (A, B)'s.

This can be done by using the Moufang identities, just as in the proof of the Triality Principle. Recall from that proof that if x is an imaginary Cayley number of unit length, then $A = L_x$, $B = -L_x$ and $C = L_x R_x$ "works", that is, $-L_x(mu) = L_x R_x(m)\, L_x(u)$. Now let x describe a semi-circular path in the i, j-plane from i to $-i$. At the beginning of the path, $C(m) = imi$, while at the end of the path $C(m) = (-i)m(-i) = imi$. Thus C describes a loop in $SO(8)$. At the beginning of the path, $(A(u), B(v)) = (iu, -iv)$, while at the end $(A(u), B(v)) = (-iu, iv)$. Hence (A, B) describes a non-loop in G. Thus G is the non-trivial double covering Spin(9) of $SO(9)$.

QED

Here is a further indication of the extent of symmetry of the Hopf fibration $H: S^7 \hookrightarrow S^{15} \to S^8$. Orient the fibres.

PROPOSITION 7.10. *Let P and Q be any two fibres of H. Then a preassigned orientation preserving rigid motion of P onto Q can be extended to a symmetry of H. In particular, the symmetries act transitively on S^{15}.*

By Lemma 7.7, the symmetries act transitively on fibres, so we may take $P = Q = L_0$. To preassign an orientation preserving rigid motion of L_0 onto itself is to preassign the map $A \in SO(8)$ in the Triality Principle, which then promises the desired symmetry of H. QED

BIBLIOGRAPHY

[Ca 1] CARTAN, Élie. Le principe de dualité et la théorie des groupes simples et semi-simples. *Bull. des Sciences Math. 49* (1925), 361-374.

[Ca 2] —— Certaines formes riemanniennes remarquables des géométries à groupe fondamental simple. *Annales École Normale 44* (1927), 345-467.

[Cu] CURTIS, C. W. The four and eight square problem and division algebras. In *Studies in modern algebra*, A. Albert ed., MAA (1963), 100-125.

[Es] ESCOBALES, R. H. Riemannian submersions with totally geodesic fibres. *J. Diff. Geom. 10* (1975), 253-276.

[Fr] FREUDENTHAL, Hans. *Oktaven, Ausnahmegruppen und Octavengeometrie.* Utrecht (1951).