

§2. Examples where the ground field is irrelevant

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **33 (1987)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **24.05.2024**

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

§ 2. EXAMPLES WHERE THE GROUND FIELD IS IRRELEVANT

Let p be a prime number (e.g. $p=2$), and let n be a natural number, $n \geq 4$.

Consider the group $H_n(p)$ defined by generators and relations:

$$H_n(p) = \langle a, b : a^{p^{n-1}} = 1, b^p = 1, ba = a^{1+p^{n-2}}b \rangle.$$

The group $H_n(p)$ is a finite p -group of order p^n .

PROPOSITION 1. *Let K be an arbitrary field of characteristic p . The group algebra $K[H_n(p)]$ does not possess any filtered multiplicative basis.*

Remark. In contrast, consider the dihedral group

$$D(2^n) = \langle r, s : r^{2^{n-1}} = 1, s^2 = 1, sr = r^{-1}s \rangle.$$

Both $D(2^n)$ and $H_n(2)$ are semi-direct products of $\mathbf{Z}/2^{n-1}\mathbf{Z}$ by $\mathbf{Z}/2\mathbf{Z}$.

However, a straightforward calculation shows that the set B consisting of the following elements

$$\begin{aligned} & 1, 1 + s, \\ & (r+s)^k, (1+s)(r+s)^k \quad \text{for } k = 1, \dots, 2^{n-2}, \\ & (r+s)^l (1+s), (1+s)(r+s)^l (1+s) \quad \text{for } l = 1, \dots, 2^{n-2} - 1 \end{aligned}$$

is a filtered multiplicative basis of $K[D(2^n)]$ for any field K of characteristic 2.

We proceed to prove Proposition 1. Let $M = \text{rad } K[H_n(p)]$. Recall that a, b are the generators of the defining presentation of $H_n(p)$.

LEMMA. *Let L_k be the set*

$$L_k = \{(1-a)^{k_1} (1-b)^{k_2} \mid 0 \leq k_1 < p^{n-1}, 0 \leq k_2 < p \text{ and } k_1 + k_2 = k\}.$$

Claim: *The classes mod M^{k+1} of the elements of L_k form a K -basis of M^k/M^{k+1} .*

Proof. We show first, by induction on k , that the set

$$\{(1-a)^l (1-b)^{k-l} \mid 0 \leq l \leq k\}$$

is a system of K -generators of $M^k \text{ mod } M^{k+1}$.

If $g, g' \in H_n(p)$, the identity

$$1 - g \cdot g' = (1-g) + (1-g') - (1-g)(1-g')$$

implies that $\{(1-a), (1-b)\}$ is a system of K -generators of $M \bmod M^2$.

Suppose by induction that

$$\{(1-a)^l (1-b)^{m-l} \mid 0 \leq l \leq m\}$$

is a K -generator system of $M^m \bmod M^{m+1}$. The set of products $u_1 \cdot u_2$ with $u_1 \in M$, $u_2 \in M^m$ generates M^{m+1} over K . Thus we have by induction,

$$u_1 = \alpha_1(1-a) + \alpha_2(1-b) \bmod M^2 \quad \text{with } \alpha_1, \alpha_2 \in K,$$

$$u_2 = \sum_{l=0}^m \beta_l(1-a)^l (1-b)^{m-l} \bmod M^{m+1} \quad \text{with } \beta_l \in K.$$

Hence

$$\begin{aligned} u_1 \cdot u_2 &= \sum_{l=0}^m (\alpha_1 \beta_l (1-a)^{l+1} (1-b)^{m-l} + \alpha_2 \beta_l (1-b) (1-a)^l (1-b)^{m-l}) \\ &\quad \bmod M^{m+2}. \end{aligned}$$

Now,

$$\begin{aligned} (1-b)(1-a) &= 1 - a - b + ba \\ &= 1 - a - b + ab - (ab - ba) \\ &= (1-a)(1-b) - (ab - ba). \end{aligned}$$

But

$$ab - ba \in M^{p^{n-2}} \subset M^3 \quad (\text{recall } n \geq 4),$$

since

$$ab - ba = ab - a^{1+p^{n-2}} b = (1-a)^{p^{n-2}} ab.$$

It follows that

$$(1-b)(1-a) = (1-a)(1-b) \bmod M^3$$

and therefore

$$(1-b)(1-a)^l (1-b)^{m-l} = (1-a)^l (1-b)^{m-l+1} \bmod M^{m+2}.$$

Consequently,

$$u_1 \cdot u_2 = \sum_{l=0}^m (\alpha_1 \beta_l (1-a)^{l+1} (1-b)^{m-l} + \alpha_2 \beta_l (1-a)^l (1-b)^{m-l+1}) \bmod M^{m+2}$$

and the set

$$L = \{(1-a)^{k_1} (1-b)^{k_2} \mid 0 \leq k_1 < p^{n-1}, 0 \leq k_2 < p\}$$

is a system of K -generators of $K[H_n(p)]$.

Since

$$|L| = p^n = |H_n(p)| = \dim_K K[H_n(p)],$$

it follows that L is a K -basis of $K[H_n(p)]$.

We have just proved that

$$L_k = \{(1-a)^{k_1} (1-b)^{k_2} \mid 0 \leq k_1 < p^{n-1}, 0 \leq k_2 < p, k_1 + k_2 = k\}$$

generates $M^k \bmod M^{k+1}$.

We have to prove that L_k is linearly free over K . If $\sum_{t \in L_k} \alpha_t t = 0 \bmod M^{k+1}$ where $\alpha_t \in K$ then we can write $\sum_{t \in L_k} \alpha_t t = \sum_{\substack{s \in UL_l \\ l > k}} \beta_s s$ where $\beta_s \in K$. Consequently $\alpha_t = 0$ for all t in L_k because L is a K -basis of $K[H_n(p)]$.

We now come to the proof that $K[H_n(p)]$ has no filtered multiplicative basis.

We proceed by contradiction. If B were such a basis, consider

$$\{u, v\} = B \cap \{M \setminus M^2\},$$

the set of elements of B in M but outside M^2 .

$\{u, v\}$ is a K -basis of $M \bmod M^2$. Also $K[H_n(p)] = K[u, v]$, the algebra generated over K by u and v .

We are going to prove:

Claim: $u \cdot v = v \cdot u$

This implies that $K[H_n(p)] = K[u, v]$ is commutative: Contradiction.

Proof of the claim. By the lemma,

$$u = x_1(1-a) + y_1(1-b) \bmod M^2$$

$$v = x_2(1-a) + y_2(1-b) \bmod M^2,$$

where $x_1, x_2, y_1, y_2 \in K$ and $x_1y_2 - x_2y_1 \neq 0$.

Now,

$$u \cdot v = x_1x_2(1-a)^2 + y_1y_2(1-b)^2 + (x_1y_2 + x_2y_1)(1-a)(1-b) \bmod M^3$$

$$v \cdot u = x_1x_2(1-a)^2 + y_1y_2(1-b)^2 + (x_1y_2 + x_2y_1)(1-a)(1-b) \bmod M^3$$

since

$$(1-a)(1-b) = (1-b)(1-a) \bmod M^3.$$

We know that $(1-a)^2, (1-b)^2, (1-a)(1-b)$ forms a K -basis of M^2/M^3 . Hence $u \cdot v \neq 0$ and $v \cdot u \neq 0 \bmod M^3$. Otherwise

$$x_1x_2 = y_1y_2 = x_1y_2 + x_2y_1 = 0$$

and $x_1y_2 - x_2y_1 = 0$ contrary to the fact that $\{u, v\}$ gives a basis of M/M^2 .

Thus $uv, vu \in B$ satisfy $uv = vu \bmod M^3$ and $uv \neq 0, vu \neq 0 \bmod M^3$.

It follows that $uv = vu$. In fact more generally, if $u_1, u_2 \in B \setminus M^k$ and $u_1 = u_2 \bmod M^k$ then $u_1 = u_2$. Proof: $B \cap M^k$ is a basis of M^k , thus $u_1 - u_2 = \sum_{u \in B \cap M^k} \lambda_u u$. This is possible only if $u_1 - u_2 = 0$.

§ 3. THE GROUP OF QUATERNION UNITS

Let Q be defined by generators and relations:

$$Q = \langle a, b : a^4 = 1, b^2 = a^2, ab = b^3a \rangle.$$

Set $i = a, j = b, k = ab$ and $c = a^2$. Then

$$Q = \{1, c, i, ci, j, cj, k, ck\}.$$

PROPOSITION 2. *Let K be a field of characteristic 2. The group algebra $K[Q]$ possesses a filtered multiplicative basis if and only if K contains a primitive cube root of unity.*

Proof. If K contains a primitive cube root of unity, say ω , let

$$B = \{1, u, v, uv, vu, u^2, v^2, u^3\},$$

where

$$u = \omega i + \omega^2 j + k$$

$$v = \omega^2 i + \omega j + k.$$

It is easily verified that B is a filtered multiplicative basis.

Conversely, suppose that $K[Q]$ possesses a filtered multiplicative basis B .