1. An isotopy closing lemma

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1. An isotopy closing lemma

We will prove the following closing lemma:

Main Lemma 1.1. *Let* $h: M \to M$ *be a homeomorphism of the connected manifold* M. *If* h *has a nonwandering point which is not a fixed point, then there exists an isotopy* $\{h_t \mid t \in [0, 1]\}$ *such that:*

(i) $h_0 = h$;

(ii) $h_t = h$ *outside a compact subset of* $M \backslash \mathrm{Fix}(h)$ *which does not depend on* t;

(iii) $\mathrm{Fix}(h_t) = \mathrm{Fix}(h)$;

(iv) h_1 *has a periodic point of period 2 in* $M \backslash \mathrm{Fix}(h_1)$.

We will need several elementary lemmas. The first lemma is easy.

Lemma 1.2. *Let* $\varphi_1, \ldots, \varphi_k$ *be homeomorphisms of a space* Z. *If* X *is a subset of* Z, *we have*

$$\varphi_k \ldots \varphi_1(X) \subset X \cup \Big(\bigcup_{i=1}^{k} \mathrm{supp}(\varphi_i)\Big).$$

Lemma 1.3. *Suppose that* h *and* $\varphi_1, \ldots, \varphi_k$ *are homeomorphisms of the space* Y. *If we have*

$$\forall i = 1, \ldots, k, h(\mathrm{supp}\, \varphi_i) \cap \Big(\bigcup_{j \leq i} \mathrm{supp}\, \varphi_j\Big) = \emptyset$$

then $\mathrm{Fix}(\varphi_k \ldots \varphi_1 h) = \mathrm{Fix}(h)$.

Proof. Since $h(\mathrm{supp}\, \varphi_i) \cap \mathrm{supp}\, \varphi_i = \emptyset$, we have

$$\mathrm{Fix}(h) \cap \Big(\bigcup_{i=1}^{k} \mathrm{supp}\, \varphi_i\Big) = \emptyset .$$

This implies the inclusion $\mathrm{Fix}(h) \subset \mathrm{Fix}(\varphi_k \ldots \varphi_1 h)$.

We prove the other inclusion by induction on k.

Suppose $k = 1$. If $\varphi_1 h(x) = x$ and $h(x) \neq x$ then certainly $h(x) \in \mathrm{supp}\, \varphi_1$ and hence also $x = \varphi_1 h(x) \in \mathrm{supp}\, \varphi_1$. But this is impossible, since $h(\mathrm{supp}\, \varphi_1) \cap \mathrm{supp}\, \varphi_1 = \emptyset$.

Suppose the lemma true for $k - 1$. Let x be such that $\varphi_k \varphi_{k-1} \ldots \varphi_1 h(x) = x$. This is equivalent to $\varphi_{k-1} \ldots \varphi_1 h(x) = \varphi_k^{-1}(x)$. If $x \notin \mathrm{supp}\, \varphi_k$, we obtain $\varphi_{k-1} \ldots \varphi_1 h(x) = x$. By the induction hypothesis, this gives $x \in \mathrm{Fix}(h)$. If

$x \in \operatorname{supp} \varphi_k$, then $h(x) \in h(\operatorname{supp} \varphi_k)$, and since $h(x) = \varphi_1^{-1} \dots \varphi_{k-1}^{-1} \varphi_k^{-1}(x)$, we obtain, by 1.2, that $h(x) \in \bigcup_{j \leqslant k} \operatorname{supp} \varphi_j$ which is disjoint from $h(\operatorname{supp} \varphi_k)$! □

The next definition is due to Brouwer.

Definition 1.4 *(Translation arc).* Let $h: Z \to Z$ be a homeomorphism of the space Z. An injective arc $\alpha \subset Z$ is called a translation arc (for h) if α joins some point x to its image $h(x)$ and $h(\alpha) \cap \dot{\alpha} = \emptyset$, where $\dot{\alpha}$ is α minus its extremities. Remark that α does not contain any of the fixed points of h. Moreover, we have $h(x) \in \alpha \cap h(\alpha)$ and if $\alpha \cap h(\alpha) \neq \{h(x)\}$ then $x = h^2(x)$.

LEMMA 1.5 (Brouwer). *Let* $h: M \to M$ *be a homeomorphism of the manifold* M. *If* y *and* $h(y)$ *are contained in the same component of* $M \setminus \operatorname{Fix}(h)$, *then there exists a translation arc* α *with* $y \in \dot{\alpha}$.

Proof (Well known). We can assume M connected and $\operatorname{Fix}(h) = \emptyset$. Let B be a subset of M homeomorphic to the euclidean closed ball of the same dimension as M, containing y in its interior and with $h(B) \cap B = \emptyset$. Since M is connected, there exists an isotopy $\{\theta_t \mid t \in [0, 1]\}$ such that $\theta_0 = \mathrm{Id}$, $\theta_t(y) = y$ and $\theta_1(h(y)) \in B$. If we put $B_t = \theta_t^{-1}(B)$, there is a first t such that $B_t \cap h(B_t) \neq \emptyset$, we call s this first t. We have:

(i) y is in the interior of B_s;

(ii) the interiors of B_s and $h(B_s)$ are disjoint;

(iii) B_s intersects $h(B_s)$ in a point which on the boundary of each one of them. If we call $h(x)$ this point, then x is also in the boundary of B_s.

It follows that we can find an arc $\alpha \subset B_s$ between x and $h(x)$, with $\dot{\alpha}$ contained in the interior of B_s. By (ii) above, $h(\alpha) \cap \dot{\alpha} = \emptyset$. □

PROPOSITION 1.6. *Let* α *be a translation arc for the homeomorphism* h *of the connected manifold* M. *If for some* $n \geqslant 2$ *we have* $h^n(\alpha) \cap \alpha \neq \emptyset$, *then there exists an isotopy* $\{h_t \mid t \in [0, 1]\}$ *such that:*

(i) $h_0 = h$;

(ii) $h_t = h$ *outside a compact subset of* $M \setminus \operatorname{Fix}(h)$ *which does not depend on* t;

(iii) $\operatorname{Fix}(h_t) = \operatorname{Fix}(h)$;

(iv) h_1 *has a periodic point of period 2 in* $M \setminus \operatorname{Fix}(h_1)$.

Proof. We call $x, h(x)$ the extremities of α. By 1.4, we are reduced to the case $\alpha \cap h(\alpha) = \{h(x)\}$. Call $n+1$ the first integer $\geqslant 2$ such that $h^{n+1}(\alpha) \cap \alpha \neq \emptyset$. Let $z \in h^{n+1}(\alpha) \cap \alpha$. By our choice of $n+1$, if $n+1 \geqslant 3$ and the fact that $\alpha \cap h(\alpha) = \{h(x)\}$, if $n+1 = 2$, we have $z \neq h(x)$. We orient the injective segment $\bigcup_{i=0}^{n} h^i(\alpha)$ from x to $h^{n+1}(x)$. We denote by $\leqslant$ the natural order induced by this orientation with $x < h(x)$. We first consider the case where $h^{-2}(z) \leqslant z$. Let $\beta \subset \alpha \backslash \{h(x)\}$ be the compact sub arc joining $h^{-2}(z)$ to z. We have $h(\beta) \cap \beta = \emptyset$. Let V be a small connected neighborhood of β such that $h(V) \cap V = \emptyset$. Call φ_t an isotopy of M with compact support contained in V and such that $\varphi_0 = \mathrm{Id}$ and $\varphi_1(z) = h^{-2}(z)$. We can define h_t as $\varphi_t h$. By 1.3, the conditions $h(V) \cap V = \emptyset$ and $\operatorname{supp}(\varphi_t) \subset V$ imply that $\operatorname{Fix}(\varphi_t h) = \operatorname{Fix}(h)$. Furthermore, since $h^{-2}(z) \in V$, we have $h^{-1}(z) \in h(V)$ which does not intersect $\operatorname{supp}(\varphi_1)$. It follows that $(\varphi_1 h)(h^{-2}(z)) = h^{-1}(z)$, and hence we obtain $(\varphi_1 h)^2(h^{-2}(z)) = \varphi_1(z) = h^{-2}(z)$.

We now consider the case $z \leqslant h^{-2}(z)$. We choose $z_0 = z \leqslant z_1 \leqslant \dots \leqslant z_k = h^{-2}(z)$ in the segment $\bigcup_{i=0}^{n-1} h^i(\alpha)$ such that the subsegment $[z_0, z_i]$ is disjoint from the image $h([z_{i-1}, z_i])$, for $i = 1, \dots, k$. We can find neighborhoods $V_1, \dots, V_i, \dots, V_k$ of $[z_0, z_1], \dots, [z_{i-1}, z_i], \dots, [z_{k-1}, z_k]$ such that $h(V_i) \cap (\bigcup_{j \leqslant i} V_j) = \emptyset$. It is easy to construct a sequence of isotopies with compact support $\varphi_t^1, \dots, \varphi_t^k$ such that $\varphi_1^i(z_{i-1}) = z_i$ and $\operatorname{supp} \varphi_t^i \subset V_i$. By 1.3, this last condition and the fact that $h(V_i) \cap (\bigcup_{j \leqslant i} V_j) = \emptyset$, for $i = 1, \dots, k$, imply the equality $\operatorname{Fix}(\varphi_t^k \dots \varphi_t^1 h) = \operatorname{Fix}(h)$. Moreover, since $h^{-1}(z) \in h(V_k)$ which is disjoint from $\bigcup_{i=1}^{k} \operatorname{supp} \varphi_i$, we have $(\varphi_1^k \dots \varphi_1^1 h)^2(h^{-2}(z)) = h^{-2}(z)$. □

COROLLARY 1.7. *Let α be a translation arc for the homeomorphism h of the connected manifold M. Suppose that some point of α is in the closure of $\bigcup_{n \geqslant 2} h^n(\alpha)$, then there exists an isotopy $\{h_t \mid t \in [0, 1]\}$ such that:*

(i) $h_0 = h$;

(ii) $h_t = h$ *outside a compact subset of* $M \backslash \operatorname{Fix}(h)$;

(iii) $\operatorname{Fix}(h_t) = \operatorname{Fix}(h)$;

(iv) h_1 *has a periodic point of period 2 in* $M \backslash \operatorname{Fix}(h_1)$.

Proof. We can suppose that $\alpha \cap (\bigcup_{n \geqslant 2} h^n(\alpha)) = \emptyset$. Then we will find an isotopy $\{h_t \mid t \in [0, 1]\}$ such that:

(i) $h_0 = h$;

(ii) α is a translation arc for each h_t;

(iii) $\operatorname{Fix}(h_t) = \operatorname{Fix}(h)$;

(iv) $h_1^n(\alpha) \cap \alpha \neq \emptyset$, for some $n \geqslant 2$;

(v) $h_t = h$ outside a compact subset of M which does not depend on t.

It will then suffice to apply proposition 1.6 to h_1.

We denote by x and $h(x)$ the extremities of α. Let us call $z \in \alpha \backslash \{h(x)\}$ a point of accumulation of $\bigcup_{n \geqslant 2} h^n(\alpha)$. Let V be a small connected neighborhood of z which does not intersect $h(\alpha)$. Let $n \geqslant 2$ be the first integer such that $h^n(\alpha)$ intersects V. We can find an isotopy $\{\varphi_t \mid t \in [0, 1]\}$, with compact support contained in V, such that $\varphi_0 = \mathrm{Id}$ and $\varphi_1 h^n(\alpha) \ni z$. It suffices to define h_t as $\varphi_t h$. □

LEMMA 1.8. *Let h be a homeomorphism of the manifold M. Suppose that h has a non-wandering point for h which is not a fixed point, then there exists an isotopy $\{h_t \mid t \in [0, 1]\}$ such that:*

(i) $h_0 = h$;

(ii) $h_t = h$ *outside a compact subset of* $M \backslash \mathrm{Fix}\,(h)$;

(iii) $\mathrm{Fix}\,(h_t) = \mathrm{Fix}\,(h)$;

(iv) *there is a periodic point of* h_1 *which is not a fixed point.*

Proof. Call z a non-wandering point which is not a fixed point. Let V be a small open connected neighborhood of z such that $h(V) \cap V = \emptyset$. Call $n \geqslant 2$ the first integer such that $h^n(V) \cap V \neq \emptyset$. Choose $y \in h^{-n}(V) \cap V \neq \emptyset$. Call $\{\varphi_t \mid t \in [0, 1]\}$ an isotopy with compact support in V and such that $\varphi_1(h^n(y)) = y$. It suffices to put $h_t = \varphi_t h$. □

Proof of the Main Lemma. If h leaves invariant each component of $M \backslash \mathrm{Fix}\,(h)$, the Main Lemma follows from what we have done. If this is not the case then by a result of Brown and Kister [BK] $M \backslash \mathrm{Fix}\,(h)$ has exactly two connected components which are exchanged by h. It is easy to construct the required isotopy in this case. □

Remarks 1.9. (i) In the proof of the Main Lemma, we use the Brown-Kister result only in the case where $\mathrm{Fix}\,(h)$ disconnects x from $h(x)$. In particular, if M is connected, of dimension $\geqslant 2$, and if $\mathrm{Fix}\,(h)$ is finite we do not have to use it.

(ii) It follows from [Bw2, Lemma 6.3] that a homeomorphism of a connected manifold of dimension $\geqslant 3$ which is not the identity can be isotoped without changing the set of fixed point to a homeomorphism with a periodic point of period 2. Hence, the main lemma 1.1 is useful only for dimension 2.