2. KÄHLER AND ALGEBRAIC STRUCTURES

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **33 (1987)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **24.05.2024**

compact manifold is the connected sum of algebraic surfaces. For nonsimply connected algebraic surfaces, it is more difficult to speculate. The basic problem is to find a way to construct complex structures. Perhaps one can ask the following question. Suppose M is a compact almost complex manifold satisfying $\chi(M) = 3\tau(M)$ and covered topologically by $\mathbf{R}^4$. (Here $\chi(M)$ is the Euler number and $\tau(M)$ is the index of M.) If every abelian subgroup of $\pi_1(M)$ is infinite cyclic, does M admit a complex structure so that M is covered holomorphically by the unit ball in $\mathbf{C}^2$? The Lefschetz theorem may be useful in the above question.

2. Kähler and algebraic structures

Let M^n be an n complex dimensional compact manifold with complex structure J. The first question is: When is J Kählerian, i.e., (M, J) admits a Kähler metric? Harvey-Lawson [H-L] gave an intrinsic characterization of the Kählerian condition if and only if M carries no positive currents which are the (1, 1)-components of boundaries. Hodge theory gives a lot of necessary conditions for complex manifolds to be Kähler. In particular, their even Betti numbers must be positive and their odd Betti numbers are even. Also, when (M, J) is Kählerian, its rational homotopy type is determined by its rational cohomology, see Deligne-Griffiths-Morgan-Sullivan [DGMS].

Now suppose M is a Kähler manifold, i.e., M has some Kählerian complex structure. When does M admit a non-Kählerian complex structure? When does M have a unique complex (or Kählerian) structure?

When $n = 2$, every compact complex surface with even first Betti number is Kählerian. (This follows from the classification of Kodaira because Miyaoka [M1] and Siu [S1] proved respectively that elliptic surfaces with even first Betti number and $K-3$ surfaces are Kählerian. From this one concludes that among the seven classes of surfaces in Kodaira's classification, the first five are Kählerian for every complex structure. The remaining two classes of surfaces have odd first Betti number and hence admit no Kähler metrics. In particular, one sees that on a Kähler surface M^2, all complex structures on M^2 are Kählerian.)

When $n \geqslant 3$, the situation is much more complicated. Calabi [Ca3] proved that there is a non-Kählerian structure on $X \times T^2_{\mathbf{C}}$, where X is a hyperelliptic curve with genus $g = 2k + 1$, $k \geqslant 0$. On the other hand, we know that the only Kählerian structures on $X \times T^2_{\mathbf{C}}$ is the standard one.

Are there non-Kählerian complex structures on compact locally irreducible Hermitian symmetric spaces which are covered by bounded domains?

Yau made the following conjecture: Suppose $M^n (n > 2)$ is a compact Kähler manifold with negative sectional curvature; then there exist a unique Kählerian complex structure. This statement is false if the condition "negative sectional curvature" is replaced by "negative bisectional curvature".

For a locally Hermitian symmetric space M^n, Calabi and Vesentini [CV] proved that $H^1(TM) = 0$ when $n \geqslant 2$. Siu [S2] partially settled Yau's conjecture by proving the following theorem: If M^n is a compact Kähler manifold with strongly negative curvature, then the Kähler structure on M is unique.

Now suppose that M is Kähler and diffeomorphic to a compact quotient D^n/Γ of the unit ball $D \subset \mathbf{C}^n$. Prior to Siu's theorem, Yau [Y1] proved that the Kähler structure on M is unique by using the following Chern number inequality:

$$(-1)^n \cdot c_1^{n-2} \cdot c_2 \geqslant \frac{(-1)^n n}{2(n+1)} \cdot c_1^n, \tag{2}$$

where $c_1(M) < 0$. The question is: When is the complex structure on M unique? This is not known for $n \geqslant 3$. The only known result is that every complex structure on M is hyperbolic in the sense of Kobayashi, i.e., there are no non-constant holomorphic maps from $\mathbf{C}$ to M.

Inequality (2) also gives the uniqueness of the Kähler structure on $\mathbf{CP}^n$. For n odd this result is due to Hirzebruch and Kodaira [HK]. We remark that in these kinds of rigidity problems, harmonic maps seem to be very useful. In particular, modifications of Siu's $\partial\bar{\partial}$-Bochner-Kodaira would hopefully be useful (see Siu [S2] and Sampson [Sa]).

For the deformation of Kähler structures to algebraic structures, we have the well-known Kodaira conjecture: Every compact Kähler manifold can be deformed to an algebraic manifold. This is known when $n = 2$; in fact, Kodaira [Ko] proved that every compact Kähler surface can be deformed to an algebraic surface. The Kodaira conjecture is not known for $n \geqslant 3$. In particular, if M^n is a non-algebraic compact Kähler manifold and TM is its holomorphic tangent bundle, is $H^1(TM) \neq 0$? Since a compact Kähler manifold with $h^{2,0} = 0$ is algebraic, a related question is: If M is Kähler, does $h^{2,0} \neq 0$ imply $H^1(TM) \neq 0$? (It is easy to construct a map from $H^{2,0}(M)$ to $H^1(T(M))$.)