# 5. Kähler-Einstein Metrics on Noncompact Manifolds


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$c_1(L) \geqslant 0$ on $M$ and $c_1(L) > 0$ outside some subvariety of $M$. Siu [S5] proved that the converse is also true under the weaker assumption that $c_1(L)$ is nonnegative everywhere and positive at some point. Thus, a manifold which satisfies (**) is Moishezon. It is also not known whether $\mathbf{CP}^n$, $n \geqslant 4$, can admit a nonstandard structure which is Moishezon. For $n = 3$, T. Peternell [Pe] proved that if $M$ is a Moishezon 3-fold which is topologically isomorphic to $\mathbf{CP}^3$, then $M$ is the standard $\mathbf{CP}^3$. His proof depends heavily on Mori's theory of extremal rays in 3-folds. One might expect that it is helpful for this problem to study rational curves in a Moishezon manifold which is a topological $\mathbf{CP}^n$.

## 5. Kähler-Einstein Metrics on Noncompact Manifolds

We now consider Kähler-Einstein metrics on complete noncompact manifolds. Let $g$ be a complete Kähler-Einstein metric on $M^n$, i.e., $R_{i\bar{j}} = cg_{i\bar{j}}$ for some constant $c$. If $c > 0$, Myer's theorem would imply $N$ is compact. Hence, $c \leqslant 0$ and $c_1(M) \leqslant 0$. In this section we consider the case $c_1(M) < 0$ and leave the case $c_1(M) = 0$ for the next section.

One would like to characterize noncompact manifolds which admit complete Kähler-Einstein metrics $g_{i\bar{j}}$ with $R_{i\bar{j}} = -g_{i\bar{j}}$. In particular, one would like to impose conditions on $M$ to guarantee the existence and uniqueness of a Kähler-Einstein metric. First of all, uniqueness always holds. That is to say, if $M$ and $N$ are complete Kähler-Einstein manifolds with $R = -1$ and $F: M \to N$ is a biholomorphism, then $F$ is an isometry. To prove this, let $g$ and $dv$ and $g'$ and $dv'$ denote the Kähler-Einstein metrics and volume forms of $M$ and $N$, respectively. If we let $\rho = \log(F^*dv'/dv)$, then $\partial\bar{\partial}\rho = -f^*\,\mathrm{Ric}' + \mathrm{Ric} = F^*g' + g$. Taking traces, we have $\Delta\rho = -n + n \cdot e^{\rho/n}$. Hence, the maximum principle implies $\rho \leqslant 0$ and $F^*dv' \leqslant dv$. Replacing $F$ by $F^{-1}$, we have $F^*dv' \geqslant dv$ and $F$ is an isometry.

Uniqueness also holds for "almost" complete Kähler-Einstein metrics with scalar curvature equal to minus one. Here, a metric $ds^2$ on $M$ is said to be almost complete if we can write $M$ as an increasing union of domains $\Omega_\alpha$ and there exist complete metrics $ds_\alpha^2$ on $\Omega_\alpha$ for each $\alpha$ such that $ds_\alpha^2$ converges to $ds^2$ on compact subsets of $M$. See Cheng-Yau [C-Y1] for details.

We now consider the existence of Kähler-Einstein metrics with negative scalar curvature. Of course, the existence of such a metric would give restrictions on the complex structure of $M$. For example, Eiseman [Ei] proved that if there exists a Hermitian metric with scalar curvature less than a negative constant on $M$, then the pseudomeasure in the sense of Eiseman is in fact a measure, that is to say, $M$ is measure hyperbolic.

In [C-Y1], Cheng and Yau obtained the existence of Kähler-Einstein metrics on a large class of noncompact manifolds. More precisely, they proved the following. Let $M^n$ be a Hermitian manifold whose Ricci tensor defines a Kähler metric whose curvature and its covariant derivatives are bounded. Then $M$ admits a Kähler-Einstein metric which is uniformly equivalent to the above metric.

If $M$ admits a Hermitian metric with strongly negative Ricci curvature and is the increasing union of relatively compact, smooth, pseudoconvex open submanifolds, then there exists a unique (up to a scalar) almost complete Kähler-Einstein metric on $M$. Moreover, this metric is complete if $M$ is complete.

In particular, there exists a complete Kähler-Einstein metric on any bounded domain in $\mathbf{C}^n$ which is the intersection of domains with $C^2$-boundaries. In the above statement, $\mathbf{C}^n$ can also be replaced by a Hermitian manifold with Ricci curvature bounded from above by a negative constant.

Mok and Yau [Mk-Y] proved that there exists a complete Kähler-Einstein metric on any bounded pseudoconvex domain in $\mathbf{C}^n$. This is the only known "canonical" metric on arbitrary bounded domains of holomorphy which is complete.

We now consider the case where the volume of $M$ is finite. In this case, the "infinity" of $M$ is very small (whereas the infinity of a bounded domain in $\mathbf{C}^n$ is quite large). The following is then conjectured: If the Ricci curvature is negative and $M$ has finite topological type, then $M$ can be compactified, that is, $M = \bar{M}/(\text{subvariety})$ for some compact Kähler manifold $\bar{M}$. In some cases, $\bar{M}$ is actually algebraic and hence $M$ is quasi-projective.

For a locally Hermitian symmetric space $M$ of finite volume, Baily and Borel [B-B], Satake [St] and Mumford [Mu] obtained (different) compactifications more or less explicitly. For these manifolds, Kähler-Einstein metrics exist. Siu and Yau [S-Y3] proved that a complete manifold, with finite volume with its curvature bounded between two negative constants, is quasi-projective.

If the above conjecture is true, then in studying Kähler manifolds with finite volume (and bounded covariant derivatives of the curvature) one need only consider $\bar{M}\backslash(D_1 \cup \cdots \cup D_k)$ where $\bar{M}$ is a compact Kähler manifold and $D_1, \ldots, D_k$ are connected divisors. If we have suitable algebraic data on how $D_i$ looks like and how $D_i$ intersects $D_j$, then one hopes that one may be able to construct Kähler-Einstein metrics on $M$. In dimension two, this is well understood. For example, suppose $C \subseteq \bar{M}^2$ is an elliptic curve and $C \cdot C < 0$. If $s$ is a section of the bundle $[C]$ and $C = \{s=0\}$ then $dv_{\bar{M}}/|s|^2(\log|s|^2)^3$ is a complete asymptotic Kähler-Einstein metric on $\bar{M}/C$ with $C$ as the cusps of the metric.

Suppose that $D$ is a divisor on a compact Kähler manifold $M$ satisfying $c_1(K+[D]) \geqslant 0$ on $\bar{M}$, $c_1(K+[D]) > 0$ on $\bar{M}\backslash D$ and $(K+[D]) - \varepsilon[D]|_D > 0$ then $\bar{M}\backslash D$ admits a Kähler-Einstein metric with finite volume. Moreover, the curvature of the metric and all of its covariant derivatives are bounded. It is not clear whether complete Kähler-Einstein metrics should have bounded curvature.

For a quasi-projective manifold $M = \bar{M}\backslash D$, a Kähler-Einstein metric always has finite volume and one can define logarithmic Chern classes $\tilde{c}_i(M, D)$. The existence of the Kähler-Einstein metric implies the following inequality for the log Chern classes $\tilde{c}_1$ and $\tilde{c}_2$:

$$(*') \qquad (-1)^n \tilde{c}_1^{n-2} \cdot \tilde{c}_2 \geqslant \frac{(-1)^n}{2(n+1)} \tilde{c}_1^n .$$

A particularly significant fact is that equality holds in (*) if the quasi-projective manifold $\bar{M}\backslash D$ is the quotient of the unit ball in $\mathbf{C}^n$.

Recall that a complex manifold is called measure hyperbolic if the Kobayashi measure is positive everywhere. Moreover, for a complete Kähler-Einstein manifold, the following inequality holds,

$$c_1 dv_{\text{Kobayashi}} \geqslant dv_{\text{Kähler-Einstein}} \geqslant c_2 dv_{\text{Caratheodory}}$$

where $c_1$ and $c_2$ are two universal positive constants. We have the following question: If the Caratheodory metric of $M$ is complete, does $M$ admit a complete Kähler-Einstein metric?

## 6. Ricci Flat Metrics on Noncompact Manifolds

We now consider Ricci flat metrics on a complete, noncompact manifold $M$. We first remark that in this case uniqueness is unknown. Even for compact manifolds, Kähler-Einstein metrics are only unique in each Kähler class. Suppose $g$ and $g'$ are two Ricci flat Kähler metrics on $M$. If they satisfy $g_{i\bar{j}} - g'_{i\bar{j}} = \partial\bar{\partial}F$ with $F$ bounded, then $g_{i\bar{j}} = g'_{i\bar{j}}$. Note that in the compact case, the above condition means that $g$ and $g'$ belong to the same Kähler class. It also may be possible to drop the condition that $F$ is bounded since there do not exist too many Ricci flat metrics.