SIMPLE PROOF OF THE MURASUGI AND KAUFFMAN THEOREMS ON ALTERNATING LINKS

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A SIMPLE PROOF OF THE MURASUGI AND KAUFFMAN THEOREMS ON ALTERNATING LINKS

by V. G. TURAEV

The aim of the present paper is to give simplified proofs of several theorems recently obtained by Murasugi and Kauffman with the help of Jones polynomials for links. These theorems settle several old conjectures of Tait on alternating link diagrams. The proofs given here follow the main lines of the proofs given in [3], [6]; however some steps are considerably simplified, including the crucial "extended dual state Lemma".

I thank Claude Weber for careful reading of a preliminary version of this paper and for valuable suggestions. I am also indebted to Pierre de la Harpe for encouraging remarks.

§ 1. INTRODUCTION

For the definition of (smooth) links in the 3-sphere, link diagrams, alternating diagrams and alternating links, the reader is referred to [3].

A link diagram is called *reduced* if there is no (smooth) circle $S^1 \subset R^2$ intersecting the diagram in exactly two points which lie near a crossing point, as in the following picture.

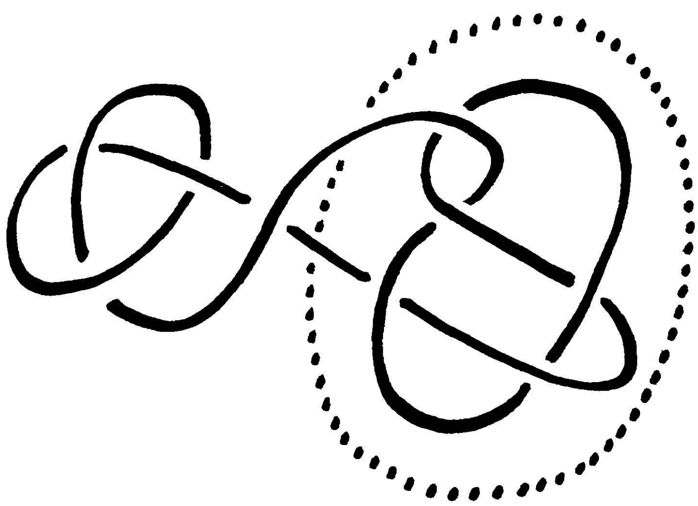


FIGURE 1

A link diagram is called *splittable* if there is a circle $S^1 \subset R^2$ which does not intersect the diagram and such that both components of $R^2 - S^1$ intersect the diagram. A link diagram K is said to be a *connected sum* of link diagrams $K_1, \ldots, K_m$ if $K_1, \ldots, K_m$ lie in disjoint discs in R^2 and if K can be obtained from $K_1, \ldots, K_m$ by band summation (the bands are supposed to lie in R^2 and to have no crossing point with each other and with $\bigcup_i K_i$). Finally, a link diagram is called *weakly alternating* if each of its split components is either a reduced alternating diagram or a connected sum of reduced alternating diagrams. Here is an example of a weakly alternating diagram which is not alternating.

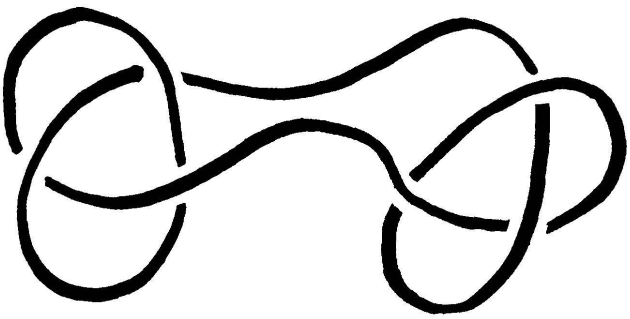


FIGURE 2

For a link diagram K we denote by $c(K)$ the *number of crossing points* of K and by $r(K)$ the *number of split components* of K.

Recall that with each oriented link $L \subset S^3$, V. Jones [4] has associated a polynomial $V_L(t) \in \mathbf{Z}[t^{1/2}, t^{-1/2}]$. If

$$V_L(t) = \sum_{n \leqslant i \leqslant m} a_i t^i \quad \text{with} \quad n, m, i \in \frac{1}{2}\mathbf{Z} \quad \text{and} \quad a_n \neq 0 \neq a_m,$$

then one defines $\operatorname{span}(L) = m - n$.

According to [4], if L has an odd number of components, then $V_L(t) \in \mathbf{Z}[t, t^{-1}]$; if L has an even number of components, then $t^{1/2}V_L(t) \in \mathbf{Z}[t, t^{-1}]$. Therefore, in all cases $\operatorname{span}(L) \in \mathbf{Z}$. Note also that $\operatorname{span}(L)$ is not changed if we invert the orientations of some components of L (thanks to the Jones reversing result, see § 8 of [3]). Thus the integer $\operatorname{span}(L)$ is an invariant of non-oriented links.

This invariant has the following additive properties. If L splits into links $L_1, \ldots, L_r$ then

$$\operatorname{span}(L) = r - 1 + \sum_{i=1}^{r} \operatorname{span}(L_i). \tag{1}$$

This follows from the formula

$$V_L(L) = (-t^{1/2}-t^{-1/2})^{r-1} \prod_{i=1}^{r} V_{L_i}(t)$$

of Jones [4]. If L is a connected sum of two links L' and L'' (performed from the unlinked union on any choice of components), then

$$V_L(L) = V_{L'}(t)V_{L''}(t)$$

so that

$$\text{span}\,(L) = \text{span}\,(L') + \text{span}\,(L'')\,.$$

THEOREM 1 (Murasugi, Kauffman). *Let K be a diagram of a link L. Then:*

(i) $c(K) + r(K) - 1 \geqslant \text{span}\,(L)$,

(ii) $c(K) + r(K) - 1 = \text{span}\,(L)$ *if and only if K is a weakly alternating diagram.*

In particular, as $r(K) = 1$ if L is unsplittable:

COROLLARY 1. *Let K be a diagram of an unsplittable link L. Then* $c(K) \geqslant \text{span}\,(L)$, *with equality if and only if K is a connected sum of reduced alternating diagrams.*

Let us observe that, if K and K' are alternating projections, one can always make connected sums K_1 and K_2 of K and K' in order that K_1 be alternating and K_2 be non-alternating. In particular, it follows that a link which has a weakly alternating projection is indeed an alternating link. See figure 3.

COROLLARY 2. *Two weakly alternating diagrams of the same alternating link L have the same number of split components. This number is equal to the number of split components of L.*

Proof. It is enough to note that every unsplittable weakly alternating diagram represents an unsplittable link. This fact is well known (at least for unsplittable alternating diagrams: see Crowell [1] and references therein). However, for the reader's convenience, we shall give here a proof of this fact which depends only on Theorem 1 and on a few elementary observations.

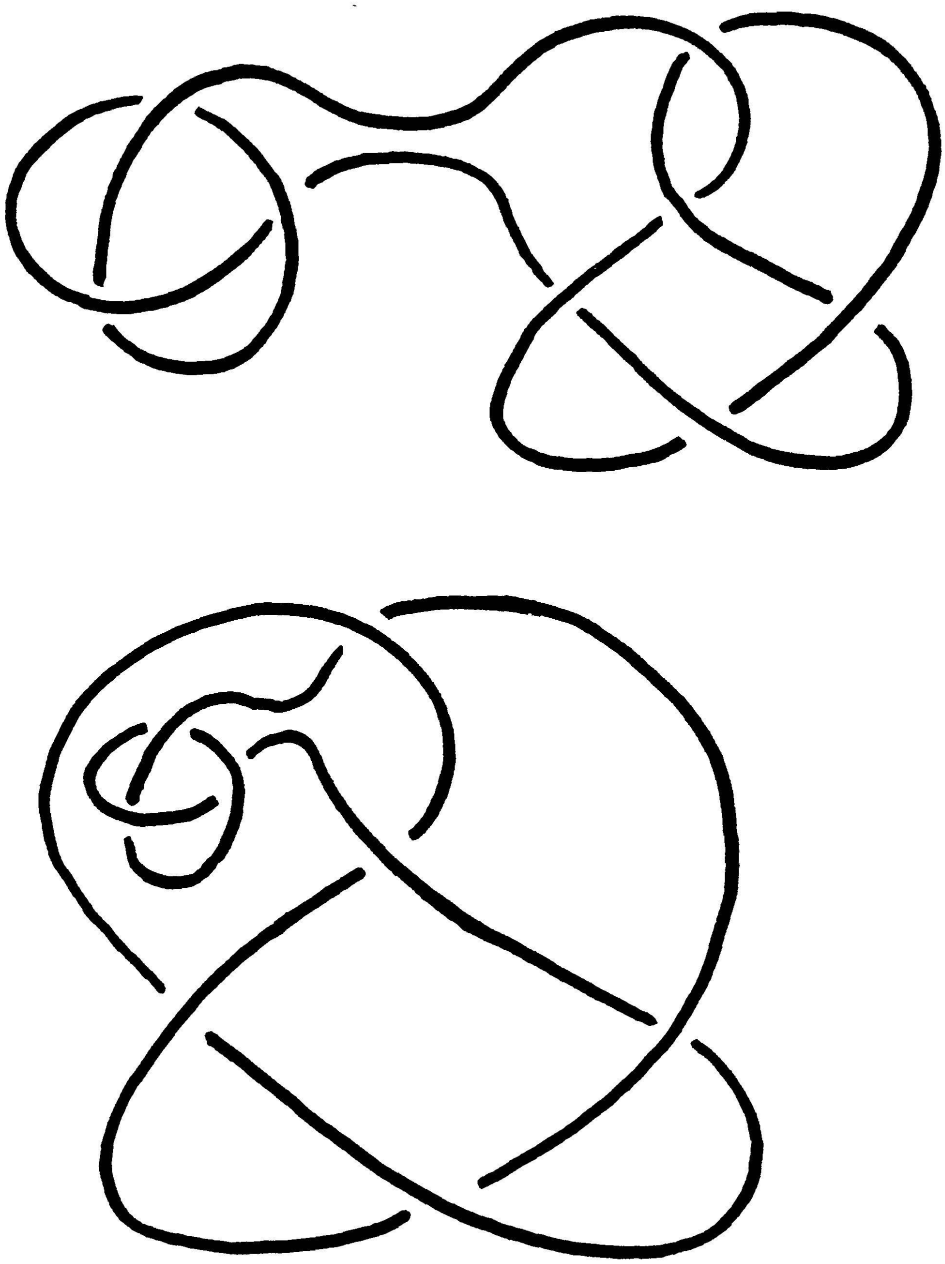


FIGURE 3

Let L be a link presented by an unsplittable weakly alternating diagram K, so that $r(K) = 1$. Suppose that L splits into unsplittable links $L_1, ..., L_p$. Then K is a "union" of subdiagrams $K_1, ..., K_p$ where K_i represents L_i for $i = 1, ..., p$. Since L_i is unsplittable, K_i is also unsplittable. In view of Corollary 1

$$(2) \qquad \sum_{i=1}^{p} c(K_i) \geqslant \sum_{i=1}^{p} \operatorname{span}(L_i) = \operatorname{span}(L) - (p-1) = c(K) - (p-1)\,.$$

Let us prove that

$$(3) \qquad c(K) \geqslant c(K_1) + c(K_2) + ... + c(K_p) + 2(p-1)\,.$$

Consider the graph with p vertices $k_1, ..., k_p$ in which the vertices k_i and k_j are connected by one edge if $i \neq j$ and if K_i crosses K_j in one point at least. Since K is unsplittable, this graph is connected. Thus it has at least $p - 1$ edges. On the other hand, the number of crossings of K_i and K_j is even for $i \neq j$. Therefore, if K_i crosses K_j at all, the number of such crossings is at least two. This implies (3).

Formulas (2) and (3) show that $p = 1$, namely that L is unsplittable. □

COROLLARY 3. *Two weakly alternating diagrams of the same alternating link L have the same number c of crossing points. This number is minimal among all diagrams of L. Any diagram of L with c crossing points is weakly alternating.*

Proof. This is straightforward from Theorem 1 and Corollary 2. Of course, $c = \operatorname{span}(L) - 1 + r$ where r is the number of split components of L. □

Remarks.

1. In the case of alternating diagrams of knots, the first two statements of Corollary 3 were conjectured by Tait [8]. For a recent discussion of this and of other conjectures by Tait, see [3].

2. For non-alternating link diagrams, the inequality (i) of Theorem 1 can be somewhat improved — see the Appendix to the present paper.

The next theorem is concerned with the writhe number of an oriented alternating link diagram. Recall that, up to isotopy in R^2, there are two types of crossing point of oriented link diagrams, distinguished by a sign:

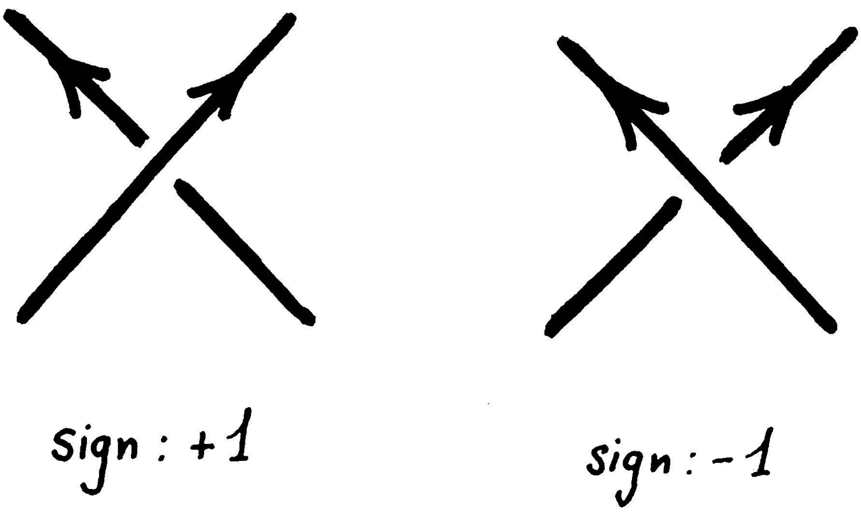


FIGURE 4

The *writhe number* $w(K)$ of an oriented link diagram K is the sum of the signs over all crossing points of K. Little believed that the writhe number of an oriented reduced alternating diagram is a link type invariant. This conjecture has been recently proved independently by Murasugi [6] and Thistlethwaite [9]. It follows directly from the following Theorem.

THEOREM 2 (Murasugi [6]). *If K is an oriented weakly alternating diagram, then*

$$w(K) = \sigma(L) - d_{\max}\big(V_L(t)\big) - d_{\min}\big(V_L(t)\big)$$

where the oriented link presented by K is denoted by L, its signature by $\sigma(L)$, and where $d_{\max}$ and $d_{\min}$ denote the maximal and minimal degrees of a polynomial. (Note that Murasugi uses the polynomial $V = V_L(t^{-1})$, so that his formula has two plus signs.)

Theorems 1 and 2 imply that, for oriented weakly alternating diagrams, both the number of positive crossing points and the number of negative crossing points are link type invariants.

It is worth realizing that, if $K^{\times}$ is the mirror image of an oriented link diagram K, then $w(K^{\times}) = -w(K)$. Therefore, if K is weakly alternating and represents an amphicheiral link, then Theorem 2 implies that $w(K) = 0$.

*
*   *

The remaining part of this paper is organized as follows. In § 2 the extended dual state Lemma, due to Kauffman and Murasugi, is stated and proved. In § 3 I quickly recall the Kauffman state model for the Jones polynomial. Theorem 1 is proved in § 4 and Theorem 2 is proved in § 5. In the Appendix, the inequality (i) of Theorem 1 is somewhat improved.

§ 2. THE EXTENDED DUAL STATE LEMMA

Let Γ be the image of a generic immersion of a finite number of circles into R^2. Note that self-crossing points of Γ are exclusively double points. For each double point x of Γ a small disc in R^2 centered in x is divided by Γ into four parts. These parts appear in two pairs of opposite sectors. Each of these pairs is called a *marker* of Γ at x. In pictures these markers are indicated like that:

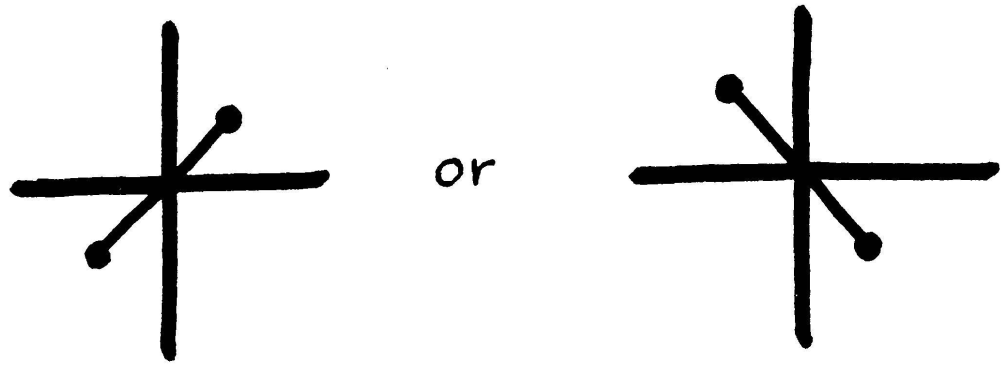


FIGURE 5

One can smooth (or surger) Γ along the markers:

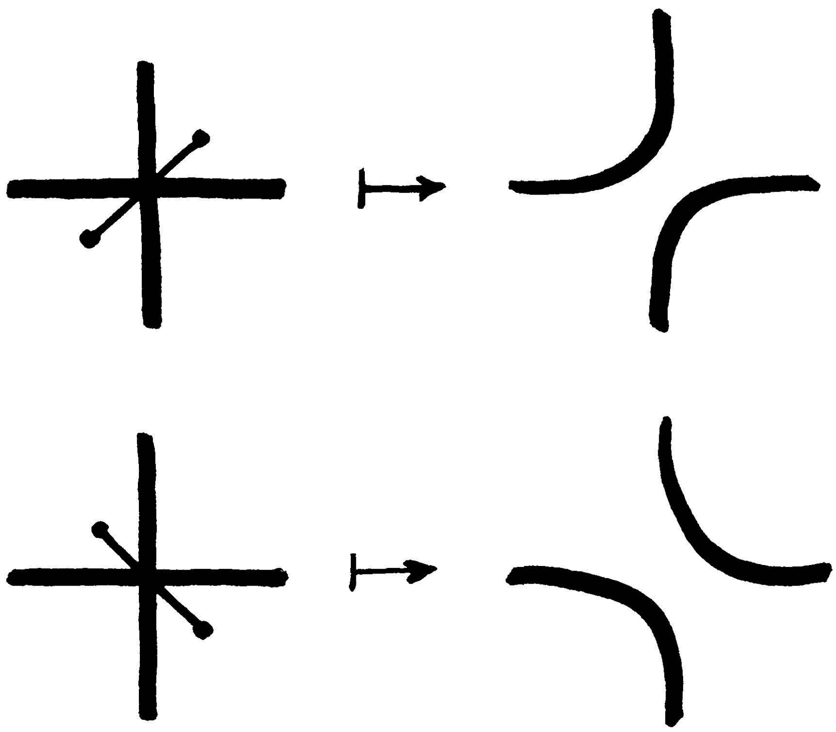


FIGURE 6

A *state* S for Γ is a choice of one marker at each double point of Γ. The opposite choice of marker at each double point defines the *dual state* of S, denoted by $\check{S}$. The dual state of $\check{S}$ is obviously S. If we surger Γ along the markers of a state S we obtain a closed imbedded 1-manifold $\Gamma_S \subset R^2$ as in the following picture.

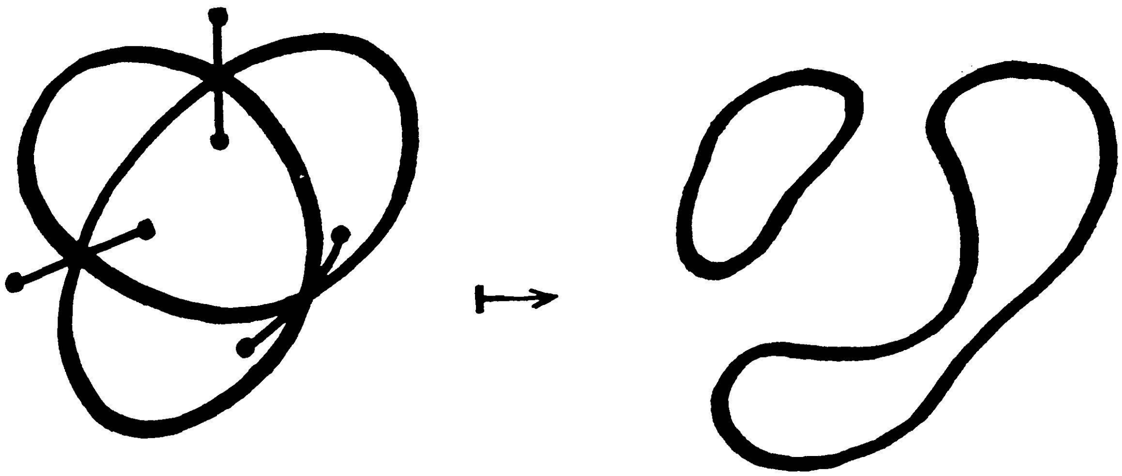


FIGURE 7

Let $|S|$ denote the number of connected components of Γ_S.

Denote by $r = r(\Gamma)$ the number of connected components of the set Γ in R^2, and by $c = c(\Gamma)$ the number of double points of Γ. It is clear that Γ has 2^c states. (If $c = 0$, then, by definition, Γ has one state S with $\Gamma_S = \Gamma$.)

LEMMA 1 (the dual state Lemma [5]). *For any state S of Γ, one has*

$$|S| + |\check{S}| \leqslant c + 2r. \tag{4}$$

To prove this Lemma and to study the case of equality in (4), we need the following definitions.

By an *edge* of Γ, we shall mean an arc in Γ whose interior does not contain any double point, and whose two ends are double points of Γ. The case of coinciding ends is not excluded, and such an edge is called a *loop*.

Let S be a state of Γ. An edge e of Γ is called *S-monochrome* if either e is a loop, or e has distinct ends and the markers of S at these ends look like this:

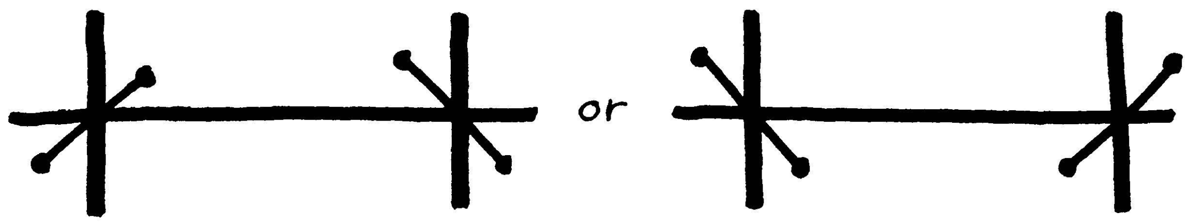


FIGURE 8

The edges of Γ which are not S-monochrome are called *S-polychrome.* Any *S-polychrome* edge has two distinct ends and the markers of S at these ends look like this:

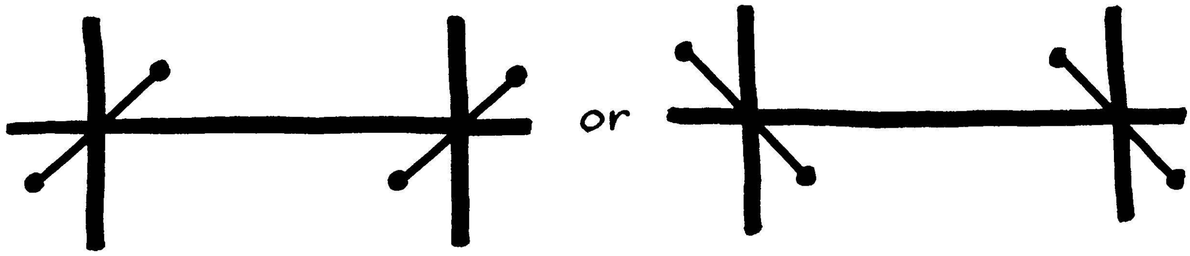


FIGURE 9

The state S of Γ is called *monochrome* if all edges of Γ are S-monochrome. It is clear that S is monochrome if and only if $\check{S}$ is monochrome.

We shall say that Γ is *prime* if each circle $S^1 \subset R^2$ which intersects Γ in exactly two points and transversally bounds a disc in $S^2 = R^2 \bigcup \{\infty\}$ which intersects Γ in a simple arc.

LEMMA 2. *Suppose that* Γ *is prime and connected. Let* S *be a state of* Γ. *Then the equality*

$$|S| + |\check{S}| = c + 2$$

holds if and only if S *is monochrome.*

Proof of lemmas 1 and 2. Let S be a state of Γ. To each double point x of Γ we associate a small square in R^2:

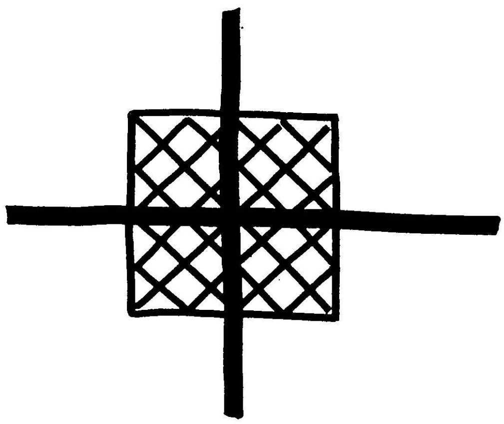


FIGURE 10

To each S-monochrome edge e of Γ we associate a plane band with core e:

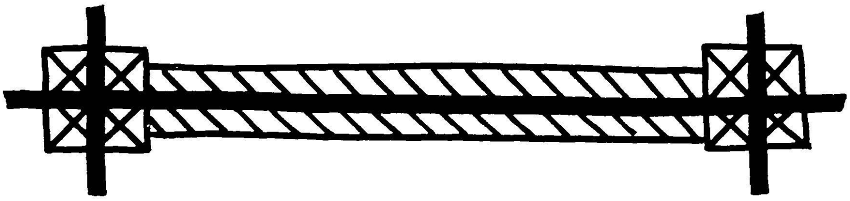


FIGURE 11

If e is a loop, the band looks like this:

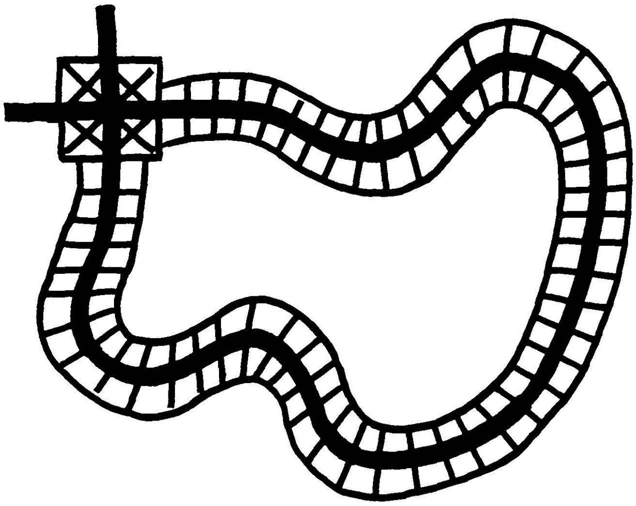


FIGURE 12

To each S-polychrome edge e we associate a 1-twisted band in R^3 with core e:

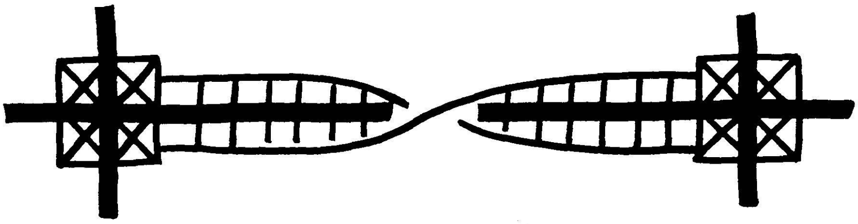
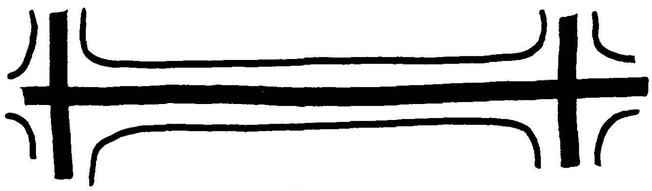


FIGURE 13

Denote by $M = M(S)$ the union of all these squares and bands. It is clear that M is a compact surface in R^3.

It is easy to check that the boundary ∂M of M is the disjoint union $\Gamma_S \amalg \Gamma_{\check{S}}$, where it is understood that Γ_S and $\Gamma_{\check{S}}$ are slightly moved away in R^3 to avoid intersections. See the following picture:



monochrome arc

polychrome arc

FIGURE 14

Therefore $|S| + |\check{S}| = b_0(\partial M)$ where b_i denote the i-th Betti number of a space with coefficients $\mathbf{Z}/2\mathbf{Z}$. As M retracts on Γ by deformation, $b_i(M) = b_i(\Gamma)$ for all i. In particular, $b_0(M) = r$. Since Γ is quadrivalent and has c double points, Γ has $2c$ edges. Thus

$$b_1(M) = b_0(M) - \chi(M) = r - (c-2c) = r + c\,.$$

Consider the homology exact sequence of the pair $(M, \partial M)$ with coefficients $\mathbf{Z}/2\mathbf{Z}$:

$$\ldots \to H_1(M) \to H_1(M, \partial M) \to H_0(\partial M) \to H_0(M) \to \{0\}\,.$$

As $b_1(M, \partial M) = b_1(M) = r + c$ by Poincaré duality, one has

$$|S| + |\check{S}| = b_0(\partial M) \leqslant b_0(M) + b_1(M, \partial M) = 2r + c\,.$$

This proves Lemma 1.

Let us now prove Lemma 2. The equality $|S| + |\check{S}| = c + 2$ holds if and only if the inclusion homomorphism $H_1(M) \to H_1(M, \partial M)$ in the exact sequence above is equal to zero. This happens if and only if the intersection form

$$H_1(M) \times H_1(M) \to \mathbf{Z}/2\mathbf{Z} \tag{5}$$

is zero. If S is monochrome then $M(S)$ is a planar surface, so that the form (5) is indeed zero.

Suppose that S is not monochrome. We shall prove that the form (5) is non zero. This will imply the strict inequality $|S| + |\check{S}| < c + 2$.

Let e be a S-polychrome edge of Γ. Consider the connected components of $R^2 - \Gamma$ which are adjacent to e. These components are distinct: Otherwise there would exist a simple loop in R^2 intersecting Γ in exactly one regular point, which is impossible. Denote these two components by a and b. It is clear that $\bar{a} \cap \bar{b}$ is a union of edges and double points of Γ, with in particular $e \in \bar{a} \cap \bar{b}$. If $\bar{a} \cap \bar{b}$ were to contain an edge of Γ distinct from e, then the dotted circle in the following picture would intersect Γ in two points.

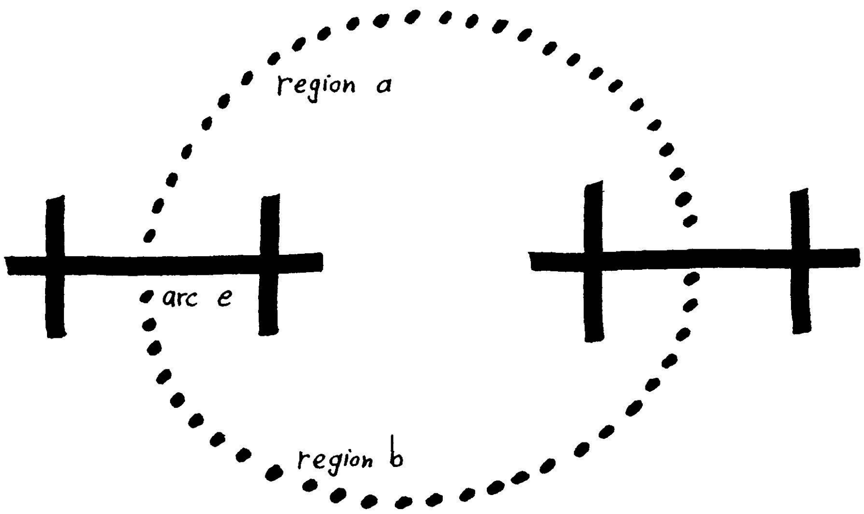


FIGURE 15

But this is impossible because Γ is prime. Thus $\bar{a} \cap \bar{b}$ is equal to the union of e and some double points.

Since e is S-polychrome, the intersection of the homology classes $[\partial \bar{a}]$ and $[\partial \bar{b}]$ in $H_1(M) \approx H_1(\Gamma)$ is equal to 1 (modulo 2):

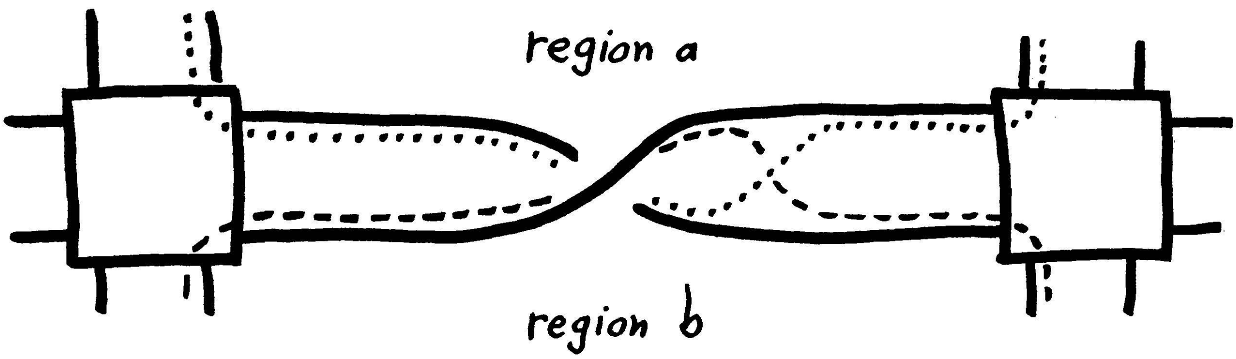


FIGURE 16

Thus (5) is a non-zero form, and the proof is complete. □

Remark. It is not important for us but curious to observe that $M(S)$ is always an orientable surface.

§ 3. KAUFFMAN'S STATE MODEL FOR THE JONES POLYNOMIAL

Let K be a link diagram. By a state or a marker of K, we mean respectively a state or a marker of the corresponding link projection in R^2 (which is obtained from K by forgetting the overcrossing-undercrossing data). The markers of K are divided into two classes — positive and negative. By definition, if the over-line is rotated counterclockwise around the double point, then the first marker it meets is the positive one and the second one is negative:

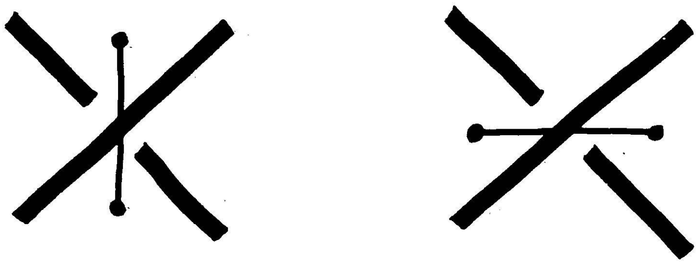


FIGURE 17

Let the diagram K be oriented. Consider the polynomial

$$V_K(t) = (-t)^{-3w(K)/4} \sum t^{(a_S - b_S)/4}(-t^{1/2} - t^{-1/2})^{|S|-1}$$

where $w(K)$ is the writhe number of K. The summation is over all the states S of K; the number of positive [respectively negative] markers of the state S is denoted by a_S [respectively b_S], and the number $|S|$ is defined in § 2.

It is shown in [5] that the polynomial $V_K(t)$ is equal to the Jones polynomial of the oriented link presented by K (see also [3]).

§ 4. PROOF OF THEOREM 1

Orient the diagram K and denote the corresponding oriented link by L. Denote by A the state of K in which all markers are positive, and by $B = \check{A}$ the dual state in which all markers are negative. For any state S of K, denote by D_S and d_S respectively the maximal and minimal degrees in t in the expression

$$t^{(a_S - b_S)/4}(-t^{1/2} - t^{-1/2})^{|S|-1}$$

(see § 3), namely

$$D_S = (a_S - b_S + 2|S| - 2)/4$$
$$d_S = (a_S - b_S - 2|S| + 2)/4\,.$$

In particular

$$D_A = (c + 2|A| - 2)/4$$

(6)

$$d_B = (-c - 2|B| + 2)/4\,.$$

Proof of (i). If a state S^2 is obtained from a state S by replacing one positive marker by a negative one (at some crossing point), then $a_{S^2} = a_S - 1$, $b_{S^2} = b_S + 1$ and $|S^2| \leqslant |S| + 1$. Thus

$$D_{S^2} - D_S = -\frac{1}{2} + (|S^2| - |S|)/2 \leqslant 0$$

so that $D_{S^2} \leqslant D_S$. This implies that $D_S \leqslant D_A$ for any state S of K. Therefore

$$d_{\max}(V_L(t)) \leqslant -\frac{3}{4}w(K) + D_A$$
$$d_{\min}(V_L(t)) \geqslant -\frac{3}{4}w(K) + d_B\,.$$

Thus in view of equalities (6) and of Lemma 1 of § 2, one has

(7) $$\begin{aligned}\operatorname{span}(L) &\leqslant D_A - d_B = (c + |A| + |B| - 2)/2\\ &\leqslant (2c + 2r - 2)/2 = c + r - 1\,.\end{aligned}$$ □

Proof of (ii). Let $K_1, \ldots, K_r$ be the unsplittable components of K, with $r = r(K)$. Denote by L_i the oriented link represented by K_i. It follows from part (i) of the Theorem and from formula (1) that

$$c(K) = \sum_{i=1}^{r} c(K_i) \geqslant \sum_{i=1}^{r} \operatorname{span}(L_i) = \operatorname{span}(L) - (r-1)\,.$$

Thus the equality $c(K) + r - 1 = \operatorname{span}(L)$ holds if and only if $c(K_i) = \operatorname{span}(L_i)$ for each i. Therefore, to prove (ii), it suffices to consider the unsplittable case $r = 1$.

It is evident that the numbers $c(K)$ and $\operatorname{span}(L)$ are both additive under connected sum of diagrams. Therefore it is enough to prove the following assertion (*).

$$(*)\quad \left\{ \begin{array}{l} \text{For a prime unsplittable diagram } K \text{ of an oriented link } L, \text{ the} \\ \text{equality } c(K) = \operatorname{span}(L) \text{ holds if and only if } K \text{ is a reduced and} \\ \text{alternating diagram.} \end{array} \right.$$

In $(*)$, note that, formally, the link L is not supposed to be prime or even unsplittable.

Suppose first that $c(K) = \operatorname{span}(L)$. Then all inequalities above are in fact equalities. As $r = 1$, one has in particular

$$|A| + |B| = c + 2r = c + 2 .$$

Lemma 2 of § 2 shows that the state A is monochrome. This implies that K is alternating, because of the easy but essential lemma:

LEMMA. *Let K be an oriented connected link diagram. Then K is alternating if and only if the state A is monochrome.*

Moreover the diagram K is reduced, since all prime diagrams are reduced except the two diagrams

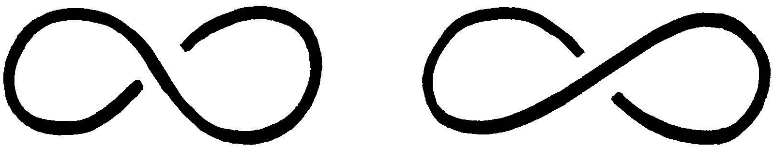


FIGURE 18

which are excluded by the assumption $c(K) = \operatorname{span}(L)$.

Suppose conversely that K is reduced and alternating. The preceeding Lemma shows that the state A is monochrome. According to Lemma 2 of § 2: $|A| + |B| = c + 2$. We prove below that

$$(8)\qquad d_{\max}(V_L(t)) = -\frac{3}{4}w(K) + D_A$$

$$(9)\qquad d_{\min}(V_L(t)) = -\frac{3}{4}w(K) + d_B .$$

Thus the inequalities (7) are in fact equalities, so that $\operatorname{span}(L) = c + r - 1 = c$.

By region, we mean hereafter a connected component of $S^2 - K$. (Here $S^2 = R^2 \cup \{\infty\}$.) Since K is alternating, each region intersects either markers which are all positive or markers which are all negative. Shade the regions of the first type:

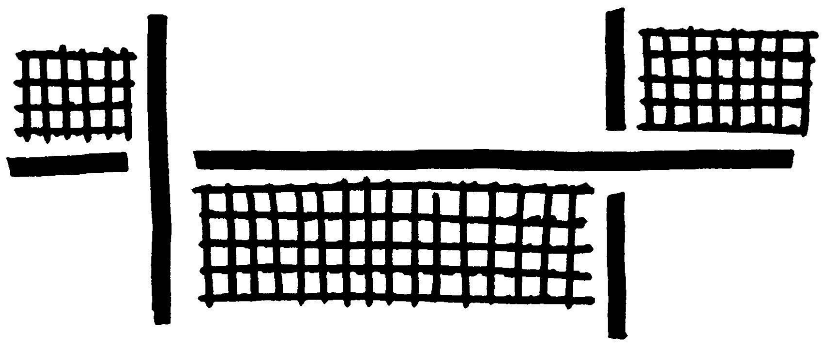


FIGURE 19

Observe that two unshaded regions near one crossing point are necessarily distinct, otherwise the diagram K would not be reduced:

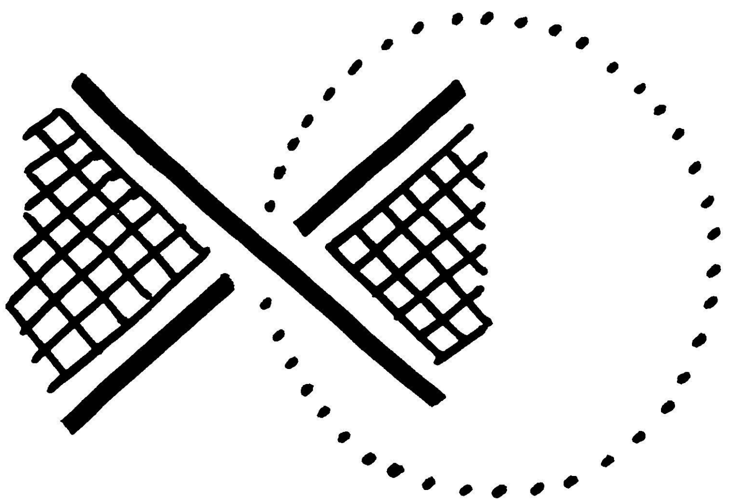


FIGURE 20

It is evident that A is equal to the number of unshaded regions. Let a state S^2 be obtained from A by replacing one positive marker by the negative marker. Under this operation two distinct unshaded regions are connected by a band, and therefore $|S^2| = |A| - 1$. In view of the arguments given in the proof of part (i) of the Theorem, this implies that $D_S < D_A$ for any state S of K. This implies (8). Analogous arguments imply (9), and the proof of (ii) in Theorem 1 is complete.

§ 5. PROOF OF THEOREM 2

Let me first recall the definition of the *signature* of an oriented link L in terms of a (not necessarily orientable) surface V bounded by L (see [2]). One defines a bilinear form

$$Q = Q_V : H_1(V; Z) \times H_1(V; Z) \to Z$$

as follows. Let $\alpha, \beta \in H_1(V; Z)$ be represented by loops a, b in V. Let us double all points of a and push them in $S^3 - V$ along both normal directions to V, at the same small distance. We obtain an oriented closed 1-manifold $\tilde{a} \in S^3 - V$; the following picture shows the local situation. The natural projection $\tilde{a} \to a$ is of course a 2-sheeted covering.

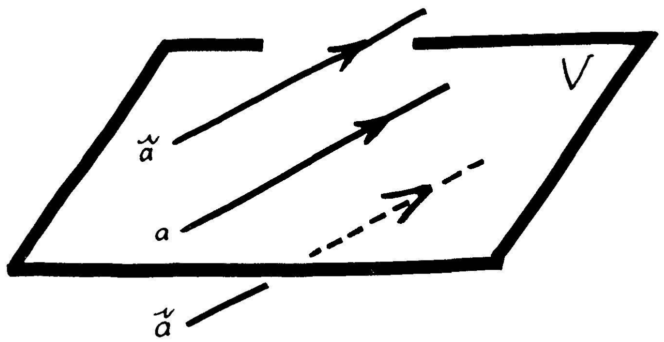


FIGURE 21

Denote by $Q(\alpha, \beta)$ the linking coefficient $Lk(\tilde{a}, b)$ of $\tilde{a}$ and b. It turns out that Q is a well defined symmetric bilinear form. Let L^V be a parallel copy of L in $S^3 - V$. Define

$$\sigma(L) = \operatorname{sign}(Q) - \frac{1}{2} Lk(L, L^V).$$

Here sign (Q) denotes the signature of the symmetric bilinear form obtained by factorizing out the annihilator of Q. According to [2], $\sigma(L)$ does not depend on the choice of the spanning surface V. In case V is orientable, $Lk(L, L^V) = 0$ and we get the classical definition of the signature of L due to Murasugi.

All diagrams and links being oriented, it is easy to check that the writhe number of a link diagram, the signature of a link, and the number $d_{\max}(V_L(t)) + d_{\min}(V_L(t))$ are additive with respect to both disjoint unions and connected sums of diagrams. Therefore it is enough to prove Theorem 2 for a diagram K which is connected, prime, alternating and reduced.

Let c_+ and c_- denote the numbers of positive and negative crossing points of such a K.

CLAIM (Murasugi). *One has* $\sigma(L) = |A| - 1 - c_+$.

This claim implies Theorem 2. Indeed, formulas (8), (9) and (6) show that

$$\begin{aligned} & d_{\max}(V_L(t)) + d_{\min}(V_L(t)) + w(K) \\ & = -w(K)/2 + D_A + d_B = -w(K)/2 + (|A|-|B|)/2 . \end{aligned}$$

Substituting in the last expression

$$w(K) = c_+ - c_-$$

$$|B| = c + 2 - |A|$$

$$c = c_+ + c_-$$

we obtain

$$\begin{aligned} & d_{\max}(V_L(t)) + d_{\min}(V_L(t)) + w(K) \\ & = |A| - 1 - c_+ = \sigma(L) . \end{aligned}$$

This implies Theorem 2.

Proof of the Claim. There is a spanning surface V of L associated with the diagram K. It is built up from shaded regions of $S^2 - K$ (see § 4) and small bands connecting these regions which enter one crossing point. In a neighbourhood of a crossing point, V looks like this:

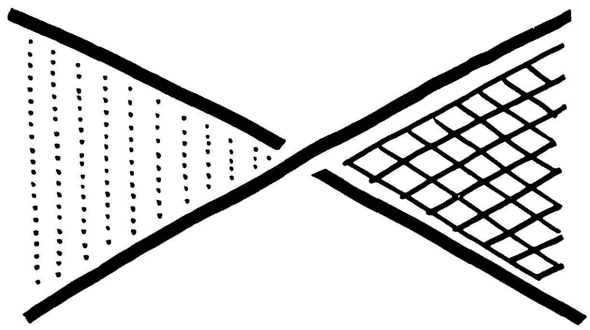


FIGURE 22

We shall prove the claim by using this surface V.

We prove first that the number $-\frac{1}{2} Lk(L, L^V)$ is equal to $-c_+$. We may assume that the push-off L^V of L in $S^3 - V$ lies in the unshaded regions of R^2 except in a neighbourhood of the crossing points. The following picture shows L^V near a crossing point (the orientations of L and L^V are not shown).

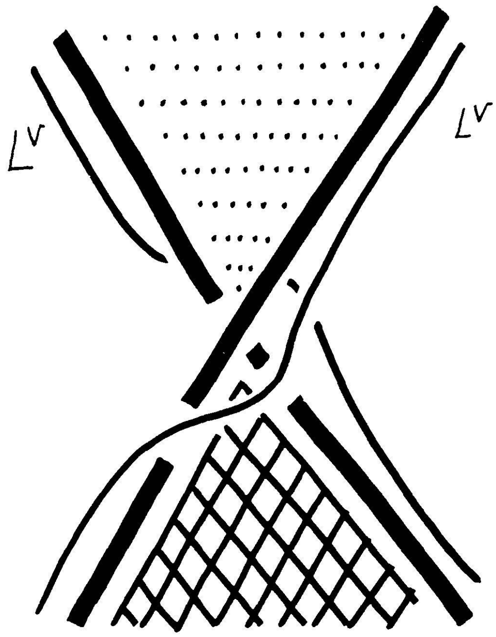


FIGURE 23

We compute $Lk(L, L^V)$, by counting the algebraic number of times L^V passes under L. It is easy to check that each crossing point of L contributes with a 2 if it is positive and with a 0 if it is negative. Thus $Lk(L, L^V) = 2c_+$.

Now, we prove that $\operatorname{sign}(Q_V) = |A| - 1$. The surface V retracts by deformation onto the complement on the unshaded regions in S^2. As the diagram is alternating, the number of unshaded regions is $|A|$, so that $b_1(V) = |A| - 1$. Thus we have to prove that the form Q_V is positive definite.

Let $\alpha \in H_1(V; Z)$ and let a be an oriented closed 1-manifold (possibly non connected) in V which represents α. Thus $Q(\alpha, \alpha) = Lk(\tilde{a}, a)$, where $\tilde{a}$ is the oriented closed 1-manifold in $S^3 - V$ obtained from a by the 2-sheeted blowing up procedure. If a subarc x of a lies in a shaded region far from crossing points of K, then, of the two corresponding subarcs of $\tilde{a}$, one lies over R^2 and the other one lies under R^2. We shall always picture the first (higher) subarc of $\tilde{a}$ on the right side of x (looking from above along a) and the second (lower) subarc of $\tilde{a}$ on the left side of x; see the following picture.

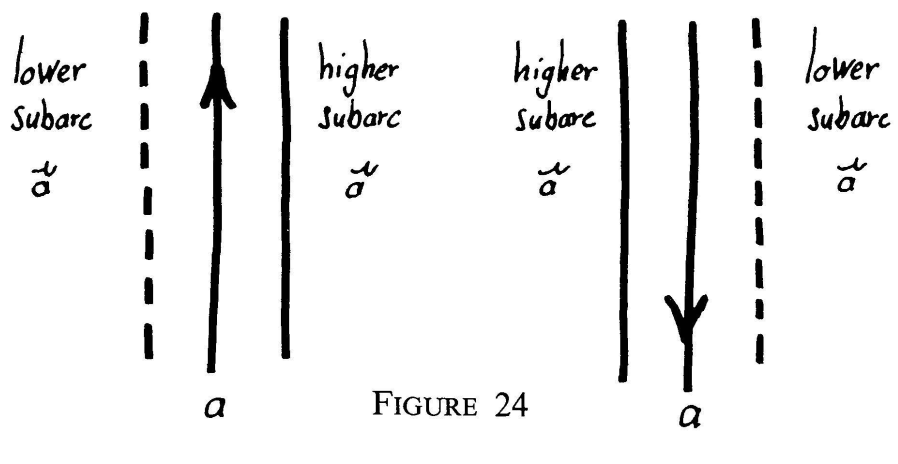


FIGURE 24

Note that the diagram of $\tilde{a}$ misses the diagram of a except in a neighborhood of the crossing points. Surgering if necessary a in V, we may assume that all components of a go through any band of V in one direction. Positions of a like those in the following picture may easily be removed by surgery.

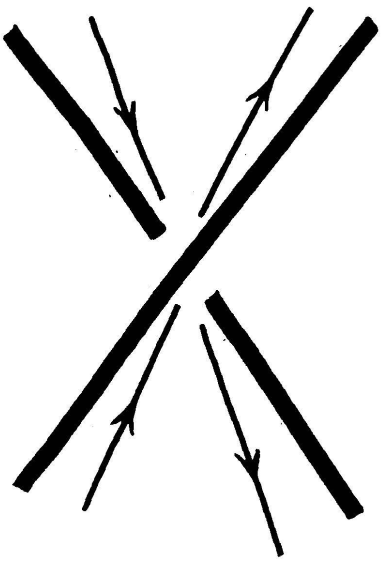


FIGURE 25

For simplicity, consider first a neighbourhood of a crossing point through which a goes only once:

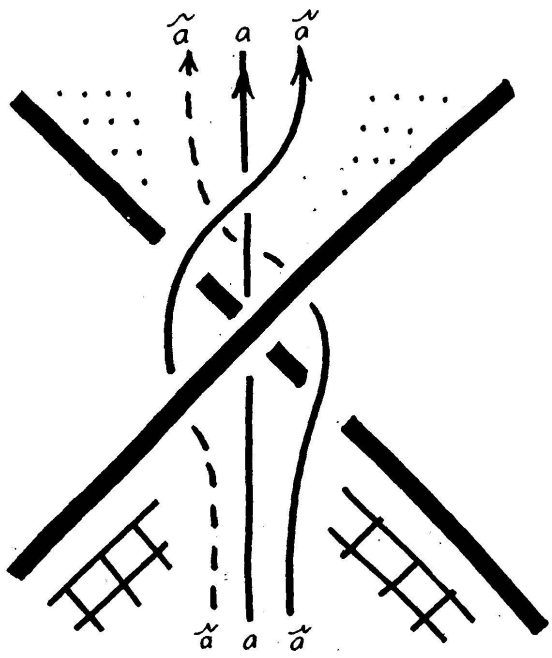


FIGURE 26

It is clear that $\tilde{a}$ passes under a in this neighbourhood one time from right to left.

If a goes through a neighbourhood $\mathscr{U}$ of a crossing point n times, then the relative positions of the corresponding n arcs of a, say $x_1, ..., x_n$, are represented as follows:

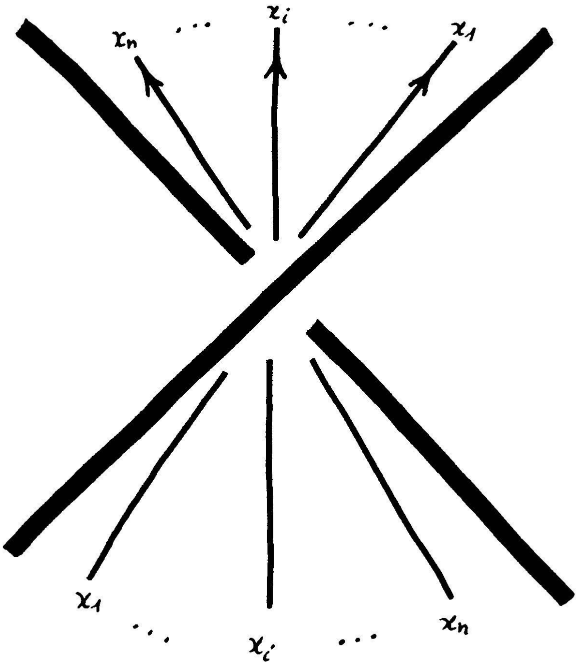


FIGURE 27

In the next picture, we show the two arcs of $\tilde{a}$ which correspond to x_i:

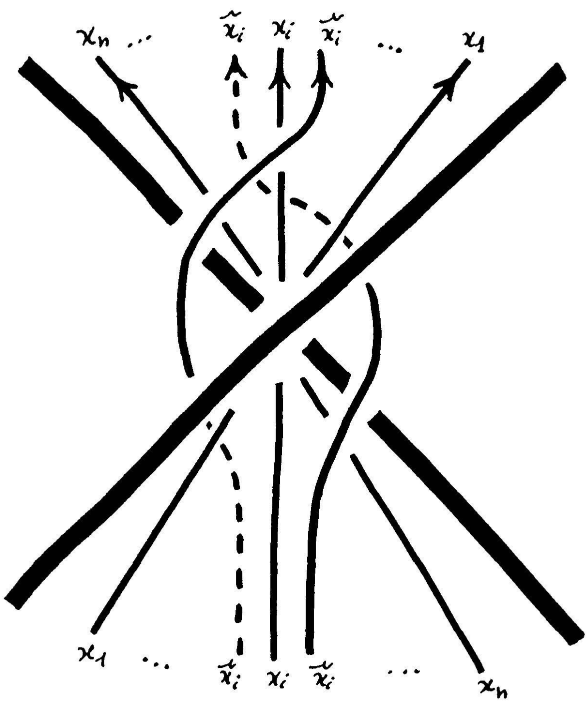


FIGURE 28

It is clear that these two arcs of $\tilde{a}$ pass $2i - 1$ times from right to left under a. Thus the contribution of the neighbourhood $\mathcal{U}$ to $Q(\alpha, \alpha)$ is given by

$$\sum_{i=1}^{n} (2i-1) = -n + 2 \sum_{i=1}^{n} i = n^2 .$$

This shows that $Q(\alpha, \alpha) > 0$ if a crosses at least one band of V. If not, then $\alpha = 0$.

Thus Q is positive definite. This completes the proof of Theorem 2.

Appendix: an improvement of the inequality of Theorem 1

Though the inequality

$$c(K) + r(K) - 1 \geqslant \operatorname{span}(L) \tag{10}$$

of Theorem 1 becomes an equality for weakly alternating diagrams, it may be sharpened a little for other cases. Let K be a link diagram in R^2 and let $\Gamma \subset R^2$ be the associated link projection. For $P \in S^2 - \Gamma$ (where $S^2 = R^2 \cup \{\infty\}$), let $i(P)$ be the intersection number modulo 2 of Γ with a generic 1-chain connecting P to ∞. Shade the regions of $S^2 - \Gamma$ for which $i \equiv 1 \pmod 2$, so that S^2 is painted like a chessboard. Let $b_1, \ldots, b_m$ be the shaded regions of $S^2 - \Gamma$ and let $w_1, \ldots, w_n$ be the unshaded regions of $S^2 - \Gamma$.

An edge e of Γ is called *K-good* either if e is a loop or if one of the end points of e corresponds to an overcrossing point of K and the other end point of e corresponds to an undercrossing point of K. An edge of Γ which is not K-good is called *K-bad.* For any $i \in \{1, \ldots, m\}$ and for any $j \in \{1, \ldots, n\}$, it is clear that the set $\overline{b_i} \cap \overline{w_j}$ consists of several edges and double points of Γ. Denote by $a(i, j)$ the number modulo 2 of K-bad edges in $\overline{b_i} \cap \overline{w_j}$. Denote by $u(K)$ the rank of the $m-by-n$ matrix $\big(a(i, j)\big)$.

Theorem. *If K is a diagram of a link L, then*

$$c(K) + r(K) - 1 \geqslant \operatorname{span}(L) + u(K) . \tag{11}$$

Corollary. *If K is a diagram of an unsplittable link L, then*

$$c(K) \geqslant \operatorname{span}(L) + u(K) .$$

Of course, if K is a weakly alternating diagram, then $u(K) = 0$.

The inequalities of the Theorem and of the Corollary may be strict. For example, if we take the diagram $K = 8_{19}$ in Rolfsen's book, then $\operatorname{span}(8_{19}) = 5$ and $u(K) = 2$, so that the inequality (11) amounts to $8 > 7$. Unfortunately, even in the case where (11) is an equality, it does not mean that K is a minimal diagram of L, since $u(K)$ depends on K and is not an invariant of L.

The proof of the Theorem goes along the same lines as the proof of Theorem 1 of § 1. Indeed the proof of Lemma 1 of § 2 shows in fact that $|S| + |\check{S}| \leqslant c + 2r - R$, where R is the rank of the intersection form (5). For the state A, it is easy to show that $R = 2u(K)$, and this gives the desired result.

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