4. SASAKI'S EQUATIONS

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **34 (1988)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **24.05.2024**

The formulas for curvature, torsion and writhe are as follows.

$$\begin{aligned} \text{Curvature} &= \kappa = \sqrt{(a^2-1)(1-b^2)} \\ \text{Torsion} &= \tau = ab \\ \text{Writhe} &= \rho = \sqrt{a^2+b^2-1}\,. \end{aligned}$$

Consider the 3-dimensional linear space of vector fields

$$aT(t) + bN(t) + cB(t)$$

which can be written as constant coefficient combinations of the Frenet vectors along the helix $p(t)$. Covariant differentiation along the helix maps this linear space to itself according to the Frenet formulas.

We've already noted in the introduction that the instantaneous axis vector $U = \tau T - \kappa B$ satisfies $U' = 0$.

Consider the vectors N and $V = (\kappa/\rho)T + (\tau/\rho)B$, which form an orthonormal basis for the orthogonal complement of U. Note that

$$\begin{aligned} N' &= -\kappa T - \tau B = -\rho V, \quad \text{and} \\ V' &= (\kappa/\rho)T' + (\tau/\rho)B' = (\kappa/\rho)(\kappa N) + (\tau/\rho)(\tau N) = \rho N\,. \end{aligned}$$

Thus, covariant differentiation along the helix kills the instantaneous axis vector and takes the orthogonal 2-plane to itself by a 90° rotation, followed by multiplication by the writhe.

4. Sasaki's equations

Let M be any Riemannian manifold, and UM its unit tangent bundle with the Riemannian metric described in section 1.

THEOREM (Sasaki [Sa], 1958). *The curve $(p(t), v(t))$ in UM is a constant speed geodesic there if and only if both of the following equations hold:*

1) $v'' = -\langle v', v'\rangle\, v$
2) $p'' = R(v', v)p'$.

Here, primes denote ordinary derivatives with respect to t when applied to functions, and covariant derivatives along the path $p(t)$ when applied to vector fields. For example, the first prime in p'' represents ordinary differentiation, the second, covariant differentiation. The symbol R denotes the Riemann curvature transformation

$$R\colon TM_p \times TM_p \to \mathrm{Hom}(TM_p, TM_p)\,.$$

We give a quick proof of Sasaki's theorem, and refer the reader interested in further details both to Sasaki's original paper and to a brief treatment of his result in [Ba-Br-Bu, pages 37-39].

First note that the energy of the curve $(p(t), v(t))$ in UM is given by

$$E = 1/2 \int_0^1 <p', p'> dt + 1/2 \int_0^1 <v', v'> dt .$$

This curve is a geodesic in UM precisely when it is a critical point of E for fixed end point variations. These include variations which fix all the foot points $p(t)$, that is, fixed end point variations of the second integral. This second integral equals the energy of the curve $u(t)$, lying in the unit sphere in the tangent space to M at $p(0)$, obtained by parallel translating $v(t)$ backwards along $p(t)$ to $p(0)$. Hence the curve $u(t)$ is a geodesic, that is, a great circle arc, in this unit sphere.

Because $u(t)$ is a unit vector field, $<u, u> = 1$. Differentiating twice, $<u'', u> + <u', u'> = 0$. Because $u(t)$ runs at constant speed along a great circle, u'' is parallel to u. Hence $u'' = - <u', u'> u$. Parallel translating this equation back out along $p(t)$, we get Sasaki's first equation.

To get Sasaki's second equation, consider a fixed end point variation $(p(t, s), v(t, s))$ of the curve $(p(t), v(t))$ in UM. Suppose this curve is a critical point of the energy E for such variations. Then

$$0 = dE/ds = 1/2 \int_0^1 \partial/\partial s <p', p'> dt + 1/2 \int_0^1 \partial/\partial s <v', v'> dt .$$

The first integrand is processed by differentiating with respect to s, then interchanging the order of the t and s differentiations, and finally setting up for integration by parts, yielding

$$\partial/\partial t <\partial p/\partial s, p'> - <\partial p/\partial s, p''> .$$

The second integrand is processed similarly, except that the Riemann curvature transformation appears as a penalty for interchanging the order of the t and s differentiations, since this time both are covariant. We get

$$\partial/\partial t <\partial v/\partial s, v'> - <\partial v/\partial s, v''> + <R(\partial p/\partial s, p')v, v'> .$$

Integrating these two expressions with respect to t, as required, the leading term of each drops out because the variation is fixed end point. Furthermore, the second term of the second expression is dead zero: since $<v, v> = 1$,

$\partial v/\partial s$ is orthogonal to v, while by Sasaki's first equation, v'' is parallel to v. We get

$$0 = \int_0^1 <\partial p/\partial s, p''> - <R(\partial p/\partial s, p')v, v'> dt .$$

Capitalizing on the symmetries of the curvature, we rewrite this as

$$0 = \int_0^1 <p'' - R(v', v)p', \partial p/\partial s> dt .$$

Since $p(t, s)$ was an arbitrary fixed end point variation, we get

$$p'' - R(v', v)p' = 0 ,$$

which is Sasaki's second equation.

Thus if the curve $(p(t), v(t))$ is a geodesic in UM, then both of Sasaki's equations must be satisfied. Conversely, if these equations are satisfied, then the curve is a critical point of the energy E for fixed end point variations, and hence a geodesic in UM. This completes the proof of Sasaki's theorem.

Here are some immediate consequences of Sasaki's theorem.

Suppose $(p(t), v(t))$ is a constant speed geodesic in UM. Then:

1) The vertical speed $| v'(t) |$ is constant. Indeed,

$$<v, v> = 1 \Rightarrow <v, v'> = 0 ,$$

and hence

$$\partial/\partial t <v', v'> = 2 <v'', v'> = -2 <v', v'> <v, v'> = 0 ,$$

by Sasaki's first equation.

2) The horizontal speed $| p'(t) |$ is also constant. We have

$$\partial/\partial t <p', p'> = 2 <p'', p'> = 2 <R(v', v)p', p'> = 0 ,$$

by Sasaki's second equation together with the skew-symmetry of the Riemann curvature tensor $<R(\cdot,\cdot)\cdot,\cdot>$ in its last two positions.

3) If $v(t)$ is a parallel vector field along $p(t)$, then Sasaki's second equation reduces to the equation $p'' = 0$ of a geodesic in M. Conversely, if $p(t)$ is a geodesic in M and $v(t)$ a parallel unit vector field along it, then Sasaki's two equations are clearly satisfied, so $(p(t), v(t))$ must be a geodesic in UM. But there will also be geodesics $(p(t), v(t))$ in UM for which $p(t)$ is a geodesic in M, while $v(t)$ is *not* parallel along $p(t)$.