

§1. Introduction

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **34 (1988)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **23.05.2024**

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

ON TORRES-TYPE RELATIONS FOR THE ALEXANDER POLYNOMIALS OF LINKS

by V. G. TURAEV

§ 1. INTRODUCTION

The classical formula of Torres [5] relates the (first) Alexander polynomial of a link K in S^3 with that of the sublink of K obtained by deleting a component. The aim of the present paper is to establish a Torres-type formula for Alexander polynomials of higher-dimensional links. We also discuss analogous formulas for higher Alexander polynomials of links in S^3 .

An n -component link in the sphere S^m is an ordered collection of n disjoint smooth imbedded oriented $(m-2)$ -dimensional spheres in S^m . With each odd-dimensional link $K \subset S^{2r+1}$ one associates a Λ_n -module $H_r(\tilde{X})$, where Λ_n is the Laurent polynomial ring $\mathbf{Z}[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}]$, X is the exterior of K and \tilde{X} is the maximal abelian covering of X . The module $H_r(\tilde{X})$ algebraically gives rise to a sequence of Fitting (or determinantal) invariants $\Delta_1(K), \Delta_2(K), \dots$, which are elements of Λ_n defined up to multiplication by monomials $\pm t_1^{s_1} \dots t_n^{s_n}$ (see [1] or § 3). The polynomial $\Delta_i(K)$ is called the i -th Alexander polynomial of K . The first Alexander polynomial $\Delta_1(K)$ is also denoted by $\Delta(K)$ and called "the Alexander polynomial of K ".

THEOREM (Torres [5]). *Let K be an n -component link in S^3 with $n \geq 2$ and let L be the sublink of K obtained by deleting the n -th component. Then*

$$\Delta(K)(t_1, \dots, t_{n-1}, 1) = \begin{cases} (t_1^{l_1} \dots t_{n-1}^{l_{n-1}} - 1)\Delta(L) & \text{if } n > 2 \\ \frac{t_1^{l_1} - 1}{t_1 - 1} \Delta(L) & \text{if } n = 2 \end{cases}$$

where l_i denotes the linking number of the i -th and n -th components of K .

The following theorem can be considered as a high-dimensional variant of the Torres theorem.

THEOREM 1. Let K be an n -component link in S^m with odd $m \geq 5$. Let L be the sublink of K obtained by deleting the n -th component. Then there exists an element λ of Λ_{n-1} such that

$$(1) \quad \Delta(L) = \Delta(K)(t_1, \dots, t_{n-1}, 1) \cdot \lambda \bar{\lambda}.$$

Here the overbar denotes the involution of the Laurent polynomial ring Λ_{n-1} which sends each polynomial $f(t_1, \dots, t_{n-1})$ into $f(t_1^{-1}, \dots, t_{n-1}^{-1})$.

It is well known that for any link $K \subset S^m$ with odd $m \geq 5$ the Alexander polynomial $\Delta(K)$ is non-zero. Moreover,

$$\text{aug}(\Delta(K)) = \Delta(K)(1, 1, \dots, 1) = \pm 1$$

(see [1]). This implies that $\text{aug}(\lambda) = \pm 1$ for any λ satisfying (1). It seems that there are no other restrictions on λ ; one may even guess that for any $\Delta \in \Lambda_n$, $\lambda \in \Lambda_{n-1}$ with $\text{aug}(\Delta) = \text{aug}(\lambda) = \pm 1$ and $\bar{\Delta} \doteq \Delta$ there exists a pair K, L as in Theorem 1 such that $\Delta(K) \doteq \Delta$ and $\Delta(L) \doteq \Delta(t_1, \dots, t_{n-1}, 1)\lambda\bar{\lambda}$. Here and below the symbol \doteq denotes the equality of Laurent polynomials up to multiplication by a monomial $\pm t_1^{s_1} \dots t_n^{s_n}$.

Let us call two Laurent polynomials $\Delta, \Delta' \in \Lambda_n$ algebraically cobordant if there exist polynomials $\lambda, \lambda' \in \Lambda_n$ such that $\Delta\lambda\bar{\lambda} \doteq \Delta'\lambda'\bar{\lambda}'$ and $\text{aug}(\lambda) = \text{aug}(\lambda') = \pm 1$. This terminology is suggested by the fact that Alexander polynomials of (smoothly) cobordant links are algebraically cobordant (see [4]). The formula (1) enables us to calculate Alexander polynomials of all sublinks of a given link, up to algebraic cobordism. It is curious to note that if K, K' are n -component links in S^m with odd $m \geq 5$ and if polynomials $\Delta(K), \Delta(K')$ are algebraically cobordant then Theorem 1 implies that Alexander polynomials of corresponding sublinks of K, K' are algebraically cobordant to each other. This fact reflects the evident property of geometric cobordisms: corresponding sublinks of cobordant links are cobordant.

I do not know if it is possible to associate with a link K some preferred $\lambda = \lambda(K)$ satisfying (1).

The remaining part of the Introduction is concerned with the classical links. The symbols $K, L, n, l_1, \dots, l_{n-1}$ denote the same objects as in the Torres theorem formulated above. It may well happen that some of the Alexander polynomials $\Delta_1(K), \Delta_2(K), \dots$ are equal to zero. Denote by $u = u(K)$ the minimal integer $u \geq 1$ such that $\Delta_u(K) \neq 0$. Since $\Delta_{i+1}(K)$ divides $\Delta_i(K)$ for all i , $\Delta_i(K) = 0$ for $i < u$ and $\Delta_i(K) \neq 0$ for $i \geq u(K)$.

In view of the Torres theorem it is natural to look for a relationship between $\Delta_{u(K)}(K)$ and a corresponding invariant of L . In the case $u(K) = 1$ we have the Torres formula, so we shall restrict ourselves to the case $u(K) \geq 2$ (i.e. the case $\Delta(K) = 0$).

The integers $u(K), u(L)$ are related by the inequality $u(L) \geq u(K) - 1$ (see [1] or § 4). If $l_i \neq 0$ at least for one $i = 1, \dots, n-1$ then the stronger inequality holds: $u(L) \geq u(K)$. These inequalities suggest to relate $\Delta_u(K)$ (where we put $u = u(K)$) with $\Delta_{u-1}(L)$ and $\Delta_u(L)$. The following relationship between $\Delta_u(K)$ and $\Delta_u(L)$ was established in [4].

THEOREM ([4, Theorem 5.5.1]). *If $u = u(K) \geq 2$ then there exist an element λ of Λ_{n-1} and a subset β of the set $\{1, 2, \dots, n-1\}$ such that*

$$(2) \quad (t_1^{l_1} \dots t_{n-1}^{l_{n-1}} - 1) \Delta_u(L) = \prod_{i \in \beta} (t_i - 1) \cdot \lambda \bar{\lambda} \cdot \Delta_u(K) (t_1, \dots, t_{n-1}, 1).$$

Several remarks are in order. a) The non-trivial case of the Theorem is the case where at least one of the integers l_1, \dots, l_{n-1} is non-zero: otherwise $t_1^{l_1} \dots t_{n-1}^{l_{n-1}} - 1 = 0$ and we may put $\lambda = 0$. b) Formula (2) is proved in [4] under the additional condition $u(L) = u(K)$. However if $u(L) < u(K)$ then we have the trivial case $l_1 = l_2 = \dots = l_{n-1} = 0$; if $u(L) > u(K)$ then $\Delta_{u(K)}(L) = 0$ and we may put $\lambda = 0$. c) Formula (2) combines the factors from the Torres formula, formula (1) and a new factor $\prod (t_i - 1)$. All these factors may be non-trivial (see [4]). d) An explicit construction of the set $\beta = \beta(K)$ is given in [4, § 5]. I do not know if there exists a preferred $\lambda = \lambda(K)$ which satisfies (2).

The relationships between the polynomials $\Delta_u(K)$ and $\Delta_{u-1}(L)$ were first considered by Levine [2] in the case $u = 2$.

THEOREM (Levine [2]). *If $u(K) \geq 2$ then there exist an element $\lambda \in \Lambda_{n-1}$ and a set $\beta \subset \{1, 2, \dots, n-1\}$ such that*

$$\Delta(L) = \prod_{i \in \beta} (t_i - 1) \cdot \lambda \bar{\lambda} \cdot \Delta_2(K) (t_1, \dots, t_{n-1}, 1).$$

Note that in the case $u(K) > 2$ the Levine's theorem is evident: if $u(K) > 2$ then $u(L) \geq u(K) - 1 > 1$ so that $\Delta(L) = \Delta_2(K) = 0$.

The following theorem generalizes the Levine's result.

THEOREM 2. *If $u = u(K) \geq 2$ then there exist an element λ of Λ_{n-1} and a set $\beta \subset \{1, 2, \dots, n-1\}$ such that*

$$\Delta_{u-1}(L) = \prod_{i \in \beta} (t_i - 1) \cdot \lambda \bar{\lambda} \cdot \Delta_u(K) (t_1, \dots, t_{n-1}, 1).$$

The non-trivial case of Theorem 2 is the case $l_1 = l_2 = \dots = l_{n-1} = 0$: otherwise $u(L) \geq u$ so that $\Delta_{u-1}(L) = 0$ and we may put $\lambda = 0$.

The proof of Theorems 1, 2 goes along the same lines as the proof of the formula (2) given in [4]. These proofs are based on a relationship between the Alexander polynomials and Reidemeister-type torsions, established in [4]. This relationship is recalled in § 2. In § 3 several easy algebraic lemmas are proved. Theorems 1, 2 are proved in § 4.

This research was completed while the author was visiting the University of Geneva. I thank the staff of the Mathematical Department of the University and especially professors J.-C. Hausmann and M. Kervaire for their hospitality.

§ 2. TORSIONS OF CHAIN COMPLEXES AND MANIFOLDS

2.1. THE TORSION OF A CHAIN COMPLEX (see [3]). Let Q be a field. If $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ are two bases of a Q -module then $a_i = \sum_{j=1}^n c_{i,j} b_j$ where $(c_{i,j})$ is a non-singular $n \times n$ -matrix over Q ; the determinant $\det(c_{i,j}) \in Q \setminus 0$ is denoted by $[a/b]$.

Let $C = (C_m \rightarrow \dots \rightarrow C_0)$ be a chain Q -complex. Suppose that each Q -module C_i is finite dimensional with a preferred basis c_i and each Q -module $H_i(C)$ also has a preferred basis h_i . (The case $C_i = 0$ or $H_i(C) = 0$ is not excluded; by definition the zero module has the empty basis.) In this setting one defines the torsion $\tau(C) \in Q$ as follows. For each $i = 1, 2, \dots, m$ choose a sequence $b_i = (b_1^i, \dots, b_{r_i}^i)$ of elements of C_i such that $\partial_{i-1}(b_i) = (\partial_{i-1}(b_1^i), \dots, \partial_{i-1}(b_{r_i}^i))$ is a basis in $\text{Im}(\partial_{i-1}: C_i \rightarrow C_{i-1})$. For each $i = 0, 1, \dots, m$ choose a lifting \tilde{h}_i of the basis h_i to $\text{Ker } \partial_{i-1}$. The combined sequence $\partial_i(b_{i+1})\tilde{h}_i b_i$ is a basis in C_i . (It is understood that $b_0 = \emptyset$ and $b_{m+1} = \emptyset$). Put

$$(3) \quad \tau(C) = \prod_{i=0}^m [\partial_i(b_{i+1})\tilde{h}_i b_i / c_i]^{\varepsilon(i)}$$

where $\varepsilon(i) = (-1)^{i+1}$. Clearly, $\tau(C) \in Q \setminus 0$. It is easy to verify that $\tau(C)$ does not depend on the choice of b_i and \tilde{h}_i .

(Note that the torsion of C defined in Milnor's survey article [3] equals $\pm \tau(C)^{-1} \in Q/\pm 1$ and that Milnor uses the additive notation for the multiplication in $Q \setminus 0 = K_1(Q)$.)

2.1.1. LEMMA (multiplicativity of torsion). Let $0 \rightarrow C' \rightarrow C \rightarrow C'' \rightarrow 0$ be a short exact sequence of m -dimensional chain complexes over a field Q .