# 2. Minimal left ideals in right topological semigroups

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **35 (1989)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **25.05.2024**

## 2. Minimal left ideals in right topological semigroups

We present in this section several well known facts which are not usually seen in early graduate courses.

2.1 Lemma (Ellis [2]). *Let $S$ be a compact Hausdorff right topological semigroup. Then $S$ has an idempotent, that is there exists $x \in S$ with $x + x = x$.*

*Proof.* Let $\mathscr{A} = \{A \subseteq S : A \neq \emptyset, A \text{ is compact, and } A + A \subseteq A\}$. Now $\mathscr{A} \neq \emptyset$ since $S \in \mathscr{A}$. Let $\mathscr{C}$ be a chain in $A$. Then $\mathscr{C}$ is a collection of closed subsets of $S$ with the finite intersection property so $\cap\mathscr{C} \neq \emptyset$ and $\cap\mathscr{C}$ is compact. Trivially $(\cap\mathscr{C}) + (\cap\mathscr{C}) \subseteq \cap\mathscr{C}$. Pick by Zorn's Lemma a minimal member $A$ of $\mathscr{A}$.

Pick $x \in A$ and let $B = A + x$. Now $B = \rho_x[A]\ (= \{\rho_x(y) : y \in A\})$ so, as the continuous image of a compact set, $B$ is compact (and trivially non-empty). Also $B + B = A + x + A + x \subseteq A + A + A + x \subseteq A + x = B$. Thus $B \in \mathscr{A}$. Since $B = A + x \subseteq A + A \subseteq A$ and $A$ is minimal, $B = A$ so $x \in B = A + x$. That is, there exists $y \in A$ with $x = y + x$.

Let $C = \{y \in A : x = y + x\}$. $\rho_x$ is continuous so $\rho_x^{-1}[\{x\}]$ is closed. Thus $C$ is closed, hence compact. To see that $C + C \subseteq C$, let $y, z \in C$. Then $y + z \in A$ and $(y+z) + x = y + (z+x) = y + x = x$ so $y + z \in C$. Thus $C \in \mathscr{A}$. Since $C \subseteq A$ and $A$ is minimal, $C = A$. Then $x \in C$ and hence $x + x = x$. $\square$

A non-empty subset $I$ of a semigroup $S$ is a left ideal if $S + I \subseteq I$, a right ideal if $I + S \subseteq I$, and a two-sided ideal if it is both a left ideal and a right ideal. It is a fact (which we will not need) that any right ideal in a compact right topological semigroup contains a minimal right ideal, which need not be closed. (For this and other interesting facts see [1].) We do need a similar fact about left ideals.

2.2 Lemma. *Let $S$ be a compact Hausdorff right topological semigroup. Any left ideal contains a minimal left ideal and minimal left ideals are closed.*

*Proof.* Let $L$ be a left ideal of $S$. Let $\mathscr{A} = \{A \subseteq L : A \text{ is a left ideal and } A \text{ is compact}\}$. Choose $x \in L$. Then $S + x = \rho_x[S]$, the continuous image of a compact space. Also $S + (S+x) = (S+S) + x \subseteq S + x$ so $S + x$ is a left ideal. Since $S + x \subseteq S + L \subseteq L$, we have $\mathscr{A} \neq \emptyset$. One easily sees that the intersection of a chain in $\mathscr{A}$ is again in $\mathscr{A}$. Choose by Zorn's Lemma a minimal member $A$ of $\mathscr{A}$.

To see that $A$ is in fact a minimal left ideal, assume we have a left ideal $B \subseteq A$ and pick $x \in B$. Then as above $S + x \in \mathscr{A}$ while $S + x \subseteq S + B \subseteq B \subseteq A$ so $S + x = A$ so $B = A$ $\square$

2.3 *Definition.* Let $S$ be a semigroup. Then $M(S) = \cup \{L: L$ is a minimal left ideal of $S\}$.

It is a fact (which we will not need) that if $S$ is a compact Hausdorff right topological semigroup, then $M(S)$ is a two-sided ideal of $S$.

2.4 LEMMA. *Let $S$ be a compact Hausdorff right topological semigroup and let $I$ be a two-sided ideal of $S$. Then $M(S) \neq \emptyset$ and $M(S) \subseteq I$.*

*Proof.* Since $S$ is a left ideal of $S$ it contains by Lemma 2.2 a minimal left ideal so $M(S) \neq \emptyset$. So see that $M(S) \subseteq I$, let $x \in M(S)$. There is a minimal left ideal $L$ of $S$ with $x \in L$. Also choose some $y \in I$. Then $y + x \in L \cap I$ (since $I$ is a right ideal) so $L \cap I \neq \emptyset$. Thus $L \cap I$ is a left ideal contained in $L$ so that $L \cap I = L$. $\square$

The proof of the following lemma is an easy exercise and we omit it.

2.5 LEMMA. *Let $S_1$ and $S_2$ be compact right topological semigroups and let $S_1 \times S_2$ have the product topology and coordinatewise operations. Then $S_1 \times S_2$ is a compact right topological semigroup. Given $x \in S_1$ and $y \in S_2$, $\lambda_x$ and $\lambda_y$ may or may not be continuous (where $\lambda_x(t) = x + t$). If $\lambda_x: S_1 \to S_1$ and $\lambda_y: S_2 \to S_2$ are continuous, then $\lambda_{(x,y)}: S_1 \times S_2 \to S_1 \times S_2$ is continuous.*

## 3. VAN DER WAERDEN'S THEOREM

We let $l \in \mathbf{N}$ be fixed throughout and show that given any finite partition of $\mathbf{N}$ some one cell contains a length $l$ arithmetic progression.

3.1 *Definition.* (a) Let $Y = (\beta\mathbf{N})^l$ with the product topology and coordinatewise operations.

(b) $E^* = \{(a, a + d, a + 2d, \ldots, a + (l-1)d): a \in \mathbf{N}$ and $d \in \mathbf{N} \cup \{0\}\}$.

(c) $I^* = \{(a, a + d, a + 2d, \ldots, a + (l-1)d): a, d \in \mathbf{N}\}$.

(d) $E = cl_Y\, E^*$

(e) $I = cl_Y\, I^*$.

Note that by Lemmas 1.1 and 2.5, $Y$ is a compact Hausdorff right topological semigroup and whenever $\mathbf{x} = (x_1, x_2, \ldots, x_l) \in \mathbf{N}^l$, $\lambda_{\mathbf{x}}$ is continuous.