

## **2. Minimal left ideals in right topological semigroups**

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## 2. MINIMAL LEFT IDEALS IN RIGHT TOPOLOGICAL SEMIGROUPS

We present in this section several well known facts which are not usually seen in early graduate courses.

**2.1 LEMMA** (Ellis [2]). *Let  $S$  be a compact Hausdorff right topological semigroup. Then  $S$  has an idempotent, that is there exists  $x \in S$  with  $x + x = x$ .*

*Proof.* Let  $\mathcal{A} = \{A \subseteq S : A \neq \emptyset, A \text{ is compact, and } A + A \subseteq A\}$ . Now  $\mathcal{A} \neq \emptyset$  since  $S \in \mathcal{A}$ . Let  $\mathcal{C}$  be a chain in  $\mathcal{A}$ . Then  $\mathcal{C}$  is a collection of closed subsets of  $S$  with the finite intersection property so  $\cap \mathcal{C} \neq \emptyset$  and  $\cap \mathcal{C}$  is compact. Trivially  $(\cap \mathcal{C}) + (\cap \mathcal{C}) \subseteq \cap \mathcal{C}$ . Pick by Zorn's Lemma a minimal member  $A$  of  $\mathcal{A}$ .

Pick  $x \in A$  and let  $B = A + x$ . Now  $B = \rho_x[A] (= \{\rho_x(y) : y \in A\})$  so, as the continuous image of a compact set,  $B$  is compact (and trivially non-empty). Also  $B + B = A + x + A + x \subseteq A + A + A + x \subseteq A + x = B$ . Thus  $B \in \mathcal{A}$ . Since  $B = A + x \subseteq A + A \subseteq A$  and  $A$  is minimal,  $B = A$  so  $x \in B = A + x$ . That is, there exists  $y \in A$  with  $x = y + x$ .

Let  $C = \{y \in A : x = y + x\}$ .  $\rho_x$  is continuous so  $\rho_x^{-1}[\{x\}]$  is closed. Thus  $C$  is closed, hence compact. To see that  $C + C \subseteq C$ , let  $y, z \in C$ . Then  $y + z \in A$  and  $(y + z) + x = y + (z + x) = y + x = x$  so  $y + z \in C$ . Thus  $C \in \mathcal{A}$ . Since  $C \subseteq A$  and  $A$  is minimal,  $C = A$ . Then  $x \in C$  and hence  $x + x = x$ .  $\square$

A non-empty subset  $I$  of a semigroup  $S$  is a left ideal if  $S + I \subseteq I$ , a right ideal if  $I + S \subseteq I$ , and a two-sided ideal if it is both a left ideal and a right ideal. It is a fact (which we will not need) that any right ideal in a compact right topological semigroup contains a minimal right ideal, which need not be closed. (For this and other interesting facts see [1].) We do need a similar fact about left ideals.

**2.2 LEMMA.** *Let  $S$  be a compact Hausdorff right topological semigroup. Any left ideal contains a minimal left ideal and minimal left ideals are closed.*

*Proof.* Let  $L$  be a left ideal of  $S$ . Let  $\mathcal{A} = \{A \subseteq L : A \text{ is a left ideal and } A \text{ is compact}\}$ . Choose  $x \in L$ . Then  $S + x = \rho_x[S]$ , the continuous image of a compact space. Also  $S + (S + x) = (S + S) + x \subseteq S + x$  so  $S + x$  is a left ideal. Since  $S + x \subseteq S + L \subseteq L$ , we have  $\mathcal{A} \neq \emptyset$ . One easily sees that the intersection of a chain in  $\mathcal{A}$  is again in  $\mathcal{A}$ . Choose by Zorn's Lemma a minimal member  $A$  of  $\mathcal{A}$ .

To see that  $A$  is in fact a minimal left ideal, assume we have a left ideal  $B \subseteq A$  and pick  $x \in B$ . Then as above  $S + x \in \mathcal{A}$  while  $S + x \subseteq S + B \subseteq B \subseteq A$  so  $S + x = A$  so  $B = A$   $\square$

2.3 *Definition.* Let  $S$  be a semigroup. Then  $M(S) = \cup \{L : L \text{ is a minimal left ideal of } S\}$ .

It is a fact (which we will not need) that if  $S$  is a compact Hausdorff right topological semigroup, then  $M(S)$  is a two-sided ideal of  $S$ .

2.4 *LEMMA.* Let  $S$  be a compact Hausdorff right topological semigroup and let  $I$  be a two-sided ideal of  $S$ . Then  $M(S) \neq \emptyset$  and  $M(S) \subseteq I$ .

*Proof.* Since  $S$  is a left ideal of  $S$  it contains by Lemma 2.2 a minimal left ideal so  $M(S) \neq \emptyset$ . So see that  $M(S) \subseteq I$ , let  $x \in M(S)$ . There is a minimal left ideal  $L$  of  $S$  with  $x \in L$ . Also choose some  $y \in I$ . Then  $y + x \in L \cap I$  (since  $I$  is a right ideal) so  $L \cap I \neq \emptyset$ . Thus  $L \cap I$  is a left ideal contained in  $L$  so that  $L \cap I = L$ .  $\square$

The proof of the following lemma is an easy exercise and we omit it.

2.5 *LEMMA.* Let  $S_1$  and  $S_2$  be compact right topological semigroups and let  $S_1 \times S_2$  have the product topology and coordinatewise operations. Then  $S_1 \times S_2$  is a compact right topological semigroup. Given  $x \in S_1$  and  $y \in S_2$ ,  $\lambda_x$  and  $\lambda_y$  may or may not be continuous (where  $\lambda_x(t) = x + t$ ). If  $\lambda_x : S_1 \rightarrow S_1$  and  $\lambda_y : S_2 \rightarrow S_2$  are continuous, then  $\lambda_{(x, y)} : S_1 \times S_2 \rightarrow S_1 \times S_2$  is continuous.

### 3. VAN DER WAERDEN'S THEOREM

We let  $l \in \mathbf{N}$  be fixed throughout and show that given any finite partition of  $\mathbf{N}$  some one cell contains a length  $l$  arithmetic progression.

3.1 *Definition.* (a) Let  $Y = (\beta\mathbf{N})^l$  with the product topology and coordinatewise operations.

- (b)  $E^* = \{(a, a + d, a + 2d, \dots, a + (l-1)d) : a \in \mathbf{N} \text{ and } d \in \mathbf{N} \cup \{0\}\}$ .
- (c)  $I^* = \{(a, a + d, a + 2d, \dots, a + (l-1)d) : a, d \in \mathbf{N}\}$ .
- (d)  $E = cl_Y E^*$
- (e)  $I = cl_Y I^*$ .

Note that by Lemmas 1.1 and 2.5,  $Y$  is a compact Hausdorff right topological semigroup and whenever  $\mathbf{x} = (x_1, x_2, \dots, x_l) \in \mathbf{N}^l$ ,  $\lambda_{\mathbf{x}}$  is continuous.