3. Van der Waerden's Theorem

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To see that A is in fact a minimal left ideal, assume we have a left ideal $B \subseteq A$ and pick $x \in B$. Then as above $S + x \in \mathscr{A}$ while $S + x \subseteq S + B \subseteq B \subseteq A$ so $S + x = A$ so $B = A$ $\square$

2.3 *Definition.* Let S be a semigroup. Then $M(S) = \cup \{L\colon L \text{ is a minimal left ideal of } S\}$.

It is a fact (which we will not need) that if S is a compact Hausdorff right topological semigroup, then $M(S)$ is a two-sided ideal of S.

2.4 LEMMA. *Let S be a compact Hausdorff right topological semigroup and let I be a two-sided ideal of S. Then $M(S) \neq \emptyset$ and $M(S) \subseteq I$.*

Proof. Since S is a left ideal of S it contains by Lemma 2.2 a minimal left ideal so $M(S) \neq \emptyset$. So see that $M(S) \subseteq I$, let $x \in M(S)$. There is a minimal left ideal L of S with $x \in L$. Also choose some $y \in I$. Then $y + x \in L \cap I$ (since I is a right ideal) so $L \cap I \neq \emptyset$. Thus $L \cap I$ is a left ideal contained in L so that $L \cap I = L$. $\square$

The proof of the following lemma is an easy exercise and we omit it.

2.5 LEMMA. *Let S_1 and S_2 be compact right topological semigroups and let $S_1 \times S_2$ have the product topology and coordinatewise operations. Then $S_1 \times S_2$ is a compact right topological semigroup. Given $x \in S_1$ and $y \in S_2$, λ_x and λ_y may or may not be continuous (where $\lambda_x(t) = x + t$). If $\lambda_x\colon S_1 \to S_1$ and $\lambda_y\colon S_2 \to S_2$ are continuous, then $\lambda_{(x,y)}\colon S_1 \times S_2 \to S_1 \times S_2$ is continuous.*

3. VAN DER WAERDEN'S THEOREM

We let $l \in \mathbf{N}$ be fixed throughout and show that given any finite partition of $\mathbf{N}$ some one cell contains a length l arithmetic progression.

3.1 *Definition.* (a) Let $Y = (\beta\mathbf{N})^l$ with the product topology and coordinatewise operations.

(b) $E^* = \{(a, a + d, a + 2d, \ldots, a + (l-1)d)\colon a \in \mathbf{N} \text{ and } d \in \mathbf{N} \cup \{0\}\}$.

(c) $I^* = \{(a, a + d, a + 2d, \ldots, a + (l-1)d)\colon a, d \in \mathbf{N}\}$.

(d) $E = cl_Y\, E^*$

(e) $I = cl_Y\, I^*$.

Note that by Lemmas 1.1 and 2.5, Y is a compact Hausdorff right topological semigroup and whenever $\mathbf{x} = (x_1, x_2, \ldots, x_l) \in \mathbf{N}^l$, $\lambda_{\mathbf{x}}$ is continuous.

3.2 LEMMA. *E is a compact Hausdorff right topological semigroup and I is a two sided ideal of E.*

Proof. Compactness is immediate and the Hausdorff property and right continuity are inherited from Y. We let $\mathbf{p} = (p_1, p_2, \dots p_l)$ and $\mathbf{q} = (q_1, q_2, \dots, q_l)$ be members of E and show that $\mathbf{p} + \mathbf{q} \in E$. We show further that if either $\mathbf{p}$ or $\mathbf{q}$ is in I, then $\mathbf{p} + \mathbf{q} \in I$.

To see that $\mathbf{p} + \mathbf{q} \in E$, let U be a neighborhood of $\mathbf{p} + \mathbf{q}$. By the continuity of $\rho_{\mathbf{q}}$, pick a neighborhood V of $\mathbf{p}$ with $V + \mathbf{q} = \rho_{\mathbf{q}}[V] \subseteq U$. Since $\mathbf{p} \in cl\, E^*$ we may pick $a \in \mathbf{N}$ and $d \in \mathbf{N} \cup \{0\}$ with

$$(a, a+d, a+2d, \dots, a+(l-1)d) \in V.$$

If $\mathbf{p} \in I$ we may presume $d \neq 0$. Let $\mathbf{x} = (a, a+d, a+2d, \dots, a+(l-1)d)$. Then $\mathbf{x} \in V$ so $\mathbf{x} + \mathbf{q} \in U$. By the continuity of $\lambda_{\mathbf{x}}$, pick a neighborhood W of $\mathbf{q}$ with $\mathbf{x} + W = \lambda_{\mathbf{x}}[W] \subseteq U$. Since $\mathbf{q} \in cl\, E^*$, pick $b \in \mathbf{N}$ and $c \in \mathbf{N} \cup \{0\}$ (with $c \neq 0$ if $\mathbf{q} \in I$) such that $(b, b+c, b+2c, \dots, b+(l-1)c) \in W$. Let $\mathbf{y} = (b, b+c, b+2c, \dots, b+(l-1)c)$. Then $\mathbf{x} + \mathbf{y} \in U \cap E^*$. If either $d \neq 0$ or $c \neq 0$, then $c + d \neq 0$ so $\mathbf{x} + \mathbf{y} \in U \cap I^*$. □

3.3 THEOREM. *Let $p \in M(\beta\mathbf{N})$ and let $\mathbf{p} = (p, p, \dots, p)$. Then $\mathbf{p} \in I$.*

Proof. We first show that $\mathbf{p} \in E$. Let $U_1 \times U_2 \times \dots \times U_l$ be a basic neighborhood of $\mathbf{p}$. Then $U_1 \cap U_2 \cap \dots \cap U_l$ is a neighborhood of p in $\beta\mathbf{N}$. Since $\mathbf{N}$ is dense, pick $a \in \mathbf{N} \cap (U_1 \cap U_2 \cap \dots \cap U_l)$. Then $(a, a, \dots, a) \in E^* \cap (U_1 \times U_2 \times \dots \times U_l)$. Thus $\mathbf{p} \in cl\, E^* = E$.

Since $p \in M(\beta\mathbf{N})$, there is a minimal left ideal L of $\beta\mathbf{N}$ with $p \in L$. Since $E + \mathbf{p}$ is a left ideal of E, pick by Lemma 2.2 a minimal left ideal L^* of E with $L^* \subset E + \mathbf{p}$. Since L^* is closed, hence compact, pick by Lemma 2.1 an idempotent $\mathbf{q} = (q_1, q_2, \dots, q_l)$ in L^*. Now $\mathbf{q} \in L^* \subseteq E + \mathbf{p}$ so pick some $\mathbf{s} = (s_1, s_2, \dots, s_l)$ in E with $\mathbf{q} = \mathbf{s} + \mathbf{p}$.

We show that $\mathbf{p} + \mathbf{q} = \mathbf{p}$. To this end let $i \in \{1, 2, \dots, l\}$. Now $q_i = s_i + p \in L$ so $\beta\mathbf{N} + q_i \subseteq \beta\mathbf{N} + L \subseteq L$. Thus $\beta\mathbf{N} + q_i$ is a left ideal contained in the minimal left ideal L so that $\beta\mathbf{N} + q_i = L$. Thus since $p \in L$ there exists $t_i \in \beta\mathbf{N}$ with $t_i + q_i = p$. But then $p + q_i = t_i + q_i + q_i = t_i + q_i = p$ as required.

Since $\mathbf{p} \in E$ and $\mathbf{q} \in L^*$, a left ideal of E, we have $\mathbf{p} = \mathbf{p} + \mathbf{q} \in L^*$ so that $\mathbf{p} \in M(E)$. Thus by Lemma 2.4, $\mathbf{p} \in I$.

3.4 COROLLARY (van der Waerden). *Let $m \in \mathbf{N}$ and let $\{A_1, A_2, \dots, A_m\}$ be a partition of $\mathbf{N}$. There exist $i \in \{1, 2, \dots, m\}$ and $a, d \in \mathbf{N}$ with $\{a, a+d, a+2d, \dots, a+(l-1)d\} \subseteq A_i$.*

Proof. By Lemma 2.4 $M(\beta\mathbf{N}) \neq \emptyset$ so pick $p \in M(\beta\mathbf{N})$ and let $\mathbf{p} = (p, p, ..., p)$. By Lemma 1.2 pick $i \in \{1, 2, ..., m\}$ such that $cl\, A_i$ is a neighborhood of p and let $U = cl\, A_i$. Then $U \times U \times ... \times U$ is a neighborhood of $\mathbf{p}$ while, by Theorem 2.3, $\mathbf{p} \in I = cl\, I^*$. Pick $a, d \in \mathbf{N}$ with $(a, a+d, a+2d, ..., a+(l-1)d) \in U \times U \times ... \times U$. Then

$$\{a, a + d, a + 2d, ..., a + (l-1)d\} \subseteq U \cap \mathbf{N} = (cl\, A_i) \cap N = A_i . \quad \square$$

We remark that if one starts with the free semigroup on l letters in place of $\mathbf{N}$, essentially the same proof yields the Hales-Jewett Theorem. See [3] for the details.

REFERENCES

[1] BERGLUND, J., H. JUNGHENN and P. MILNES. *Compact right topological semigroups and generalizations of almost periodicity.* Lecture Notes in Math. 663, Springer-Verlag, Berlin (1978).

[2] ELLIS, R. *Lectures on Topological Dynamics.* Benjamin, New York, 1969.

[3] FURSTENBERG, H. and Y. KATZNELSON. Idempotents and coloring theorems. *To appear.*

[4] FURSTENBERG, H. and B. WEISS. Topological dynamics and combinatorial number theory. *J. d'Analyse Math. 34* (1978), 61-85.

[5] GRAHAM, R. and B. ROTHSCHILD. A short proof of van der Waerden's Theorem. *Proc. Amer. Math. Soc. 42* (1974), 385-386.

[6] HINDMAN, N. Ultrafilters and Ramsey Theory — an update. *Proceedings of "Set Theory and its Applications – Conference at York".* To appear.

[7] SHELAH, S. Primitive recursive bounds for van der Waerden numbers. *J. Amer. Math. Soc. 1* (1989), 683-697.

[8] TAYLOR, A. A note on van der Waerden's Theorem. *J. Comb. Theory (Series A) 33* (1982), 215-219.

[9] VAN DER WAERDEN, B. Beweis einer Baudetschen Vermutung. *Nieuw Arch. Wisk (3) 19* (1927), 212-216.

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Vitaly Bergelson

Department of Mathematics
Ohio State University
Columbus, OH 43210
(USA)

Hillel Furstenberg

Department of Mathematics
Hebrew University
Jerusalem (Israel)

Neil Hindman

Department of Mathematics
Howard University
Washington, DC 20059
(USA)

Yitzhak Katznelson

Department of Mathematics
Stanford University
Stanford, CA 94305
(USA)