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A SIMPLE NASH-MOSER IMPLICIT FUNCTION THEOREM

by Xavier SAINT RAYMOND

This paper is devoted to the so-called "Nash-Moser implicit function theorem", a very powerful method which during the last decades helped to resolve several difficult problems of solvability for nonlinear partial differential equations (see e.g. Nash [7], Sergeraert [10], Zehnder [11], Hörmander [2]... and others!); unfortunately, the proofs that are commonly available (Moser [6], Schwartz [8], Sergeraert [10], Zehnder [11], Hörmander [2, 3, 4], Hamilton [1]) are very long and technical, and rather frightening for the uninitiated reader.

To correct this impediment, we present here a simple statement and a simple proof of this type of result, but it should be considered merely as an introduction to the subject. Indeed, the result is neither new nor optimal, and the interested reader would benefit by studying more elaborate versions such as that of Hörmander [2, 3, 4]. However, owing to this simple goal, we have been able to write a proof which avoids the use of too many parameters (usually found in such a proof) and shows more clearly the key ideas.

Let us first informally present the problem. One wants to solve an equation

$$F(u) = f$$

where F involves the variables x, the unknown function $u(x)$ and its derivatives up to the order m. If one can construct a solution u_0 of $F(u_0) = f_0$ for an f_0 close to f, the problem can be rewritten as an implicit function problem by considering $\phi(u, f) = F(u) - f$ which vanishes at (u_0, f_0). It is then sufficient to prove that the equation $\phi(v, g) = 0$ defines v as a function of g in a neighborhood of (u_0, f_0) large enough to contain f.

To prove implicit function theorems in infinite dimensional spaces (as spaces of functions usually are), one commonly uses iterative schemes. The simplest one is known as Picard's iterative scheme: ψ being a right inverse of $(\partial\phi/\partial u)(u_0, f_0)$, one sets

$$v_k = -\psi\phi(u_k, f), \quad u_{k+1} = u_k + v_k .$$

To prove the convergence of such a scheme, one needs estimates for the sequence v_k; but if one can estimate s derivatives of v_k (symbolically denoted by $|v_k|_s$), one gets estimates for only $s - m$ derivatives of $\phi(u_{k+1}, f)$ since ϕ involves the derivatives of u_{k+1} up to the order m; thus the convergence will hold by induction only if the right inverse satisfies an estimate

$$|\psi(\varphi)|_s \leqslant C |\varphi|_{s-m}$$

which is true only for a very special type of equations, namely the elliptic equations.

However, it is known that other types of equations are also solvable (for example hyperbolic equations), and for these, the previous scheme would give a recurrence estimate of the form

$$|v_{k+1}|_s \leqslant \frac{1}{2} |v_k|_{s+d}$$

with a positive shift d in the number of derivatives that are controlled, so that this scheme is not convergent. To overcome this difficulty, Nash [7] proposed another scheme involving smoothing operators so that $|v_k|_s$ could be estimated inductively for a fixed s; since this improvement came at the cost of introducing large constants, it was also required to find a scheme with much faster convergence. We won't describe here Nash's scheme nor its improvements by Hörmander [2, 3, 4], but only notice that such complicated schemes are needed if one is interested in optimal results with respect to the regularity of the solution: without their help, the function f on the right side must be very smooth in order to obtain some smoothness of the solution u.

In the theorem stated below, we will establish a C^∞ existence theorem so that the number of derivatives that are used (provided that it is finite) does not matter. For this reason, we are going to use the much simpler scheme proposed by Moser [5] which consists alternately in using Newton's scheme and Nash's smoothing operators: $\psi(u)$ being a right inverse of $(\partial\phi/\partial u)(u, f)$ and S_k being a sequence of smoothing operators closer and closer to the identity, one sets

$$v_k = -\psi(u_k)\phi(u_k, f), \qquad u_{k+1} = u_k + S_k v_k .$$

The key to get the estimates for v_k is to have at one's disposal estimates of linear growth type (see estimate (3) below), which are called "tame estimates" in Hamilton [1]; such estimates are now classical for ϕ itself or for its derivatives, but the main problem in the applications of this Nash-Moser method is to prove them also for the right inverse ψ; here, we will assume that these estimates hold (cf. (1) and (3)). In the proof we will also use the simple interpolation formula of Sergeraert [9] who introduced it to prove that Moser's scheme could lead to C^∞ results as well.

After the proof of the theorem, we propose a short description of the classical application of this type of result to the problem of isometric embedding of Riemannian manifolds, but this is given merely as an illustration of the method, and we refer to Hörmander [3], Sections 2 and 5, for the details.

To complete this lengthy introduction, we confess that the result stated here is probably the worst that can be found in the literature on the subject with respect to the number of derivatives that are used. One reason is that we have taken all the shifts in the number of derivatives equal to the maximum, d, to avoid the multiplicity of parameters. In specific applications however, it is obvious that this can be much improved. Throughout the paper, we consider the expression $\phi(u, f)$ for various u, but always the same $f \in C^\infty$ so that it can be written $\phi(u)$ as well; finally, we recall that the function u defined in the open subset Ω of $\mathbf{R}^n$ belongs to the Sobolev space $H^s(\Omega)$ if all its derivatives up to order s are square integrable over Ω (see also the definition with the Fourier transform in the appendix when $\Omega = \mathbf{R}^n$). We can now state the result.

THEOREM. *Let* $\phi : H^\infty(\mathbf{R}^n) \to H^\infty(\Omega)$ *where* Ω *is an open subset of* $\mathbf{R}^n$; *one denotes by* $|\ |_s$ *the norm in* $H^s(\mathbf{R}^n)$ *and by* $\|\ \|_s$ *the norm in* $H^s(\Omega)$. *One assumes that there exist* $u_0 \in H^\infty(\mathbf{R}^n)$, *an integer* $d > 0$, *a real number* δ *and constants* C_1, C_2 *and* $(C_s)_{s \geqslant d}$ *such that for any* $u, v, w \in H^\infty(\mathbf{R}^n)$,

$$
(1) \qquad | u - u_0 |_{3d} < \delta \Rightarrow \begin{cases} \forall s \geqslant d, \quad \| \phi(u) \|_s \leqslant C_s(1 + |u|_{s+d}) \\ \| \phi'(u)v \|_{2d} \leqslant C_1 | v |_{3d} \\ \| \phi''(u)(v, w) \|_{2d} \leqslant C_2 | v |_{3d} | w |_{3d} . \end{cases}
$$

(when one deals with (nonlinear) partial differential equations of order m, *these estimates classically hold for* $d > m + \frac{n}{2}$*). Moreover, one assumes that for every* $u \in H^\infty(\mathbf{R}^n)$ *such that* $| u - u_0 |_{3d} < \delta$, *there exists an operator* $\psi(u) : H^\infty(\Omega) \to H^\infty(\mathbf{R}^n)$ *satisfying for any* $\varphi \in H^\infty(\Omega)$,

$$
(2) \qquad \phi'(u)\psi(u)\varphi = \varphi \ \textit{in}\ \Omega, \quad \textit{and}
$$

$$(3) \qquad \forall s \geqslant d, \ |\psi(u)\varphi|_s \leqslant C_s(\|\varphi\|_{s+d} + |u|_{s+d}\|\varphi\|_{2d})$$

(the so-called "tame estimate"). Then, if $\|\phi(u_0)\|_{2d}$ *is sufficiently small (with respect to some upper bound of* $1/\delta, |u_0|_D$ *and* $(C_s)_{s \leqslant D}$ *where* $D = 16d^2 + 43d + 24$ *- sic!), there exists a function* $u \in H^\infty(\mathbf{R}^n)$ *such that* $\phi(u) = 0$ *in* Ω.

Remark. This theorem is stated with the Sobolev spaces $H^\infty(\Omega) = \bigcap_{s \geqslant 0} H^s(\Omega)$ and $H^\infty(\mathbf{R}^n)$ to be used in local solvability problems for nonlinear partial differential equations, but one can replace these spaces by gradations of Banach spaces B_s and $\mathbf{B}_s$ respectively with norms $|\ |_s$ and $\|\ \|_s$ if there exist some smoothing operators $(S_\theta)_{\theta > 1} : B_\infty \to B_\infty$ satisfying for every $v \in B_\infty$, $\theta > 1$ and s and $t \geqslant 0$

$$(4) \qquad \begin{cases} |S_\theta v|_s \leqslant C_{s,t}\theta^{s-t}|v|_t \text{ if } s \geqslant t; \\ |v - S_\theta v|_s \leqslant C_{s,t}\theta^{s-t}|v|_t \text{ if } s \leqslant t \end{cases}$$

(the construction of such smoothing operators in the case of Sobolev spaces is given in the appendix); we will also assume that $|v|_s \leqslant |v|_t$ whenever $s \leqslant t$. Actually, we will only use the operators S_{θ_k} where the sequence of real numbers θ_k is defined in the following way: $\theta_0 \geqslant 2$ to be chosen, then $\theta_{k+1} = \theta_k^{5/4}$; here are the properties of this sequence that we will use

$$(5) \qquad \begin{cases} \theta_k^5 = \theta_{k+1}^4, \text{ and} \\ \theta_0 \geqslant 2 \Rightarrow \sum_{j \geqslant 0} \theta_j^{-3} < \theta_0^{-1}; \end{cases}$$

indeed, $(5/4)^j \geqslant 1 + (j/4)$ implies $\theta_j = \theta_0^{(5/4)^j} \geqslant \theta_0^{1+(j/4)}$, then $\sum_{j \geqslant 0} \theta_j^{-3} \leqslant \theta_0^{-3}(1-\theta_0^{-3/4})^{-1} < \theta_0^{-1}$ since $\theta_0^{-2} < 1 - \theta_0^{-3/4}$ when $\theta_0 \geqslant 2$.

The solution u of the theorem will be obtained as the limit of the sequence u_k that is constructed in the following lemma.

LEMMA 1. *With the same assumptions as in the theorem and with the smoothing operators* S_{θ_k} *of the remark, the sequences*

$$v_k = -\psi(u_k)\phi(u_k), \qquad u_{k+1} = u_k + S_{\theta_k} v_k$$

are well defined for sufficiently large θ_0 *if* $\|\phi(u_0)\|_{2d} \leqslant \theta_0^{-4}$; *more precisely, there exist constants* $(U_t)_{t \geqslant d}$ *and* V *(independent of* k*) such that for* $k \geqslant 0$,

$(i)_k$ $\qquad |u_k - u_0|_{3d} < \delta$ *and* $\|\phi(u_k)\|_{2d} \leqslant \theta_k^{-4}$;

$(ii)_k$ $\qquad |v_k|_{3d+3} \leqslant V\theta_k^{-3}$;

$(iii)_k$ $\qquad \forall t \geqslant d, \quad (1+|u_{k+1}|_{t+2d}) \leqslant U_t\theta_k^{2d}(1+|u_k|_{t+2d})$

Proof. Since the property (i) implies that the sequences u_k and v_k are well defined (the operator $\psi(u)$ exists by assumption if $|u - u_0|_{3d} < \delta$), it is sufficient to prove (i), (ii) and (iii), and this is going to be done by induction. The property $(\mathrm{i})_0$ is true by assumption.

Proof of (ii). The tame estimate (3) gives for every $t \geqslant d$

$$|v_k|_t \leqslant C_t\big(\|\phi(u_k)\|_{t+d} + |u_k|_{t+d}\|\phi(u_k)\|_{2d}\big). \tag{6}$$

For $t = d$ and using $(\mathrm{i})_k$ one gets

$$|v_k|_d \leqslant C_d(1+|u_k-u_0|_{2d}+|u_0|_{2d})\;\|\phi(u_k)\|_{2d} \leqslant V_0\theta_k^{-4} \tag{7}$$

where $V_0 = C_d(1+\delta+|u_0|_{2d})$. The estimate (ii) will be obtained by interpolation between (7) and an estimate

$$|v_k|_T \leqslant V_1\theta_k^N \tag{8}$$

for a large T. To prove (8), we can use the first assumption (1) to estimate $\phi(u_k)$ in (6); this gives

$$\begin{aligned} |v_k|_t &\leqslant C_t\big(C_{t+d}(1+|u_k|_{t+2d})+|u_k|_{t+d}C_{2d}(1+|u_k-u_0|_{3d}+|u_0|_{3d})\big)\\ &\leqslant C_t\big(C_{t+d}+C_{2d}(1+\delta+|u_0|_{3d})\big)\,(1+|u_k|_{t+2d}). \end{aligned} \tag{9}$$

We now fix the values $N = 4(2d+1)$ and $T = 3d + 3 + (2d+3)(N+3)$. The estimate

$$(1+|u_j|_{T+2d}) \leqslant (1+|u_0|_{T+2d})\theta_j^N \tag{10}$$

obviously holds for $j = 0$; moreover, if it holds for some $j < k$, we get from $(\mathrm{iii})_j$ and (5)

$$\begin{aligned} (1+|u_{j+1}|_{T+2d}) &\leqslant U_T\theta_j^{2d}(1+|u_0|_{T+2d})\theta_j^{4(2d+1)}\\ &= (U_T\theta_j^{-1})\,(1+|u_0|_{T+2d})\theta_{j+1}^{4(2d+1)} \end{aligned}$$

so that (10) holds by induction for $j \leqslant k$ if one takes $\theta_0 \geqslant U_T$. Thus one gets (8) by replacing $|u_k|_{T+2d}$ in (9) by the estimate (10) for $j = k$; note that V_1 depends only on $|u_0|_{T+2d}$ and the constants C.

With $\bar\theta_k = \theta_k^{1/(2d+3)}$, the interpolation formula can now be written as

$$\begin{aligned} |v_k|_{3d+3} &\leqslant |S_{\bar\theta_k}v_k|_{3d+3} + |v_k - S_{\bar\theta_k}v_k|_{3d+3}\\ &\leqslant C_{3d+3,d}\bar\theta_k^{2d+3}|v_k|_d + C_{3d+3,T}\bar\theta_k^{3d+3-T}|v_k|_T\\ &\leqslant C_{3d+3,d}V_0\theta_k^{-3} + C_{3d+3,T}V_1\theta_k^{-3} \end{aligned}$$

because of (7), (8) and our choice of T, and this is $(\mathrm{ii})_k$.

Proof of (iii). It follows essentially from the estimate (9) above, if one observes that $| u_{k+1} |_{t+2d}$ can be estimated in terms of $| v_k |_t$ because of the relations (4): indeed, since $u_{k+1} = u_k + S_{\theta_k} v_k$,

$$| u_{k+1} |_{t+2d} \leqslant | u_k |_{t+2d} + | S_{\theta_k} v_k |_{t+2d} \leqslant | u_k |_{t+2d} + C_{t+2d,t} \theta_k^{2d} | v_k |_t$$

whence (iii) with constants U_t depending only on $| u_0 |_{3d}$ and the constants C by using (9).

Proof of (i). Since $u_k - u_0 = \sum_{j<k} S_{\theta_j} v_j$, (ii)$_j$ for $j \leqslant k$ allows us to write

$$\forall t \in [0, 1], | u_k + t S_{\theta_k} v_k - u_0 |_{3d} \leqslant \sum_{j \leqslant k} | S_{\theta_j} v_j |_{3d} \leqslant C_{3d,3d} \sum_{j \leqslant k} | v_j |_{3d}$$
$$\leqslant C_{3d,3d} V \sum_{j \leqslant k} \theta_j^{-3}.$$

By (5), $\sum_{j \geqslant 0} \theta_j^{-3} < \theta_0^{-1}$ so that we have for $\theta_0 \geqslant C_{3d,3d} V / \delta$

(11) $$\forall t \in [0, 1], | u_k + t S_{\theta_k} v_k - u_0 |_{3d} < \delta ;$$

for $t = 1$, this gives $| u_{k+1} - u_0 |_{3d} < \delta$ (first part of (i)$_{k+1}$).

To get an estimate for $\phi(u_{k+1})$, we write the following Taylor formula

$$\phi(u_{k+1}) = \phi(u_k) + \phi'(u_k) S_{\theta_k} v_k + \int_0^1 (1-t) \phi''(u_k + t S_{\theta_k} v_k)(S_{\theta_k} v_k, S_{\theta_k} v_k) dt ;$$

since $v_k = -\psi(u_k)\phi(u_k)$, (2) gives $\phi(u_k) = \phi'(u_k)(-v_k)$ in Ω, whence

$$\phi(u_{k+1}) = \varphi_1 + \varphi_2 \quad \text{with}$$

$$\varphi_1 = \phi'(u_k)(S_{\theta_k} v_k - v_k) \text{ and } \varphi_2 = \int_0^1 (1-t) \phi''(u_k + t S_{\theta_k} v_k)(S_{\theta_k} v_k, S_{\theta_k} v_k) dt .$$

Thanks to (11), we can use (1) to estimate φ_1 and φ_2: with (4) and (ii)$_k$ one gets

$$\| \varphi_1 \|_{2d} \leqslant C_1 | S_{\theta_k} v_k - v_k |_{3d} \leqslant C_1 C_{3d,3d+3} \theta_k^{-3} | v_k |_{3d+3}$$
$$\leqslant C_1 C_{3d,3d+3} V \theta_k^{-6}$$
$$\| \varphi_2 \|_{2d} \leqslant C_2 | S_{\theta_k} v_k |_{3d}^2 \leqslant C_2 C_{3d,3d}^2 | v_k |_{3d}^2 \leqslant C_2 C_{3d,3d}^2 V^2 \theta_k^{-6}$$

whence

$$\| \phi(u_{k+1}) \|_{2d} \leqslant C_0 \theta_k^{-6} = (C_0 \theta_k^{-1}) \theta_{k+1}^{-4}$$

(cf. (5)) where C_0 depends only on V and the constants C; for $\theta_0 \geqslant C_0$, we thus get (i)$_{k+1}$. The proof of the lemma is complete.

The estimates (i) and (ii) in lemma 1 give the existence of a solution $u \in H^{3d+3}(\mathbf{R}^n)$ of the equation $\phi(u) = 0$. But actually, the proof of property (ii) can be modified to prove an estimate for $| v_k |_s$ for every $s \geqslant d$.

LEMMA 2. *There exist constants* $(V_s)_{s \geqslant d}$ *such that the sequence* v_k *of lemma 1 satisfies for every* $k \geqslant 0$ *and* $s \geqslant d$

$$| v_k |_s \leqslant V_s \theta_k^{-3}.$$

Proof. Keeping the value $N = 4(2d+1)$, we get from (iii) and (5) that

$$(1+|u_{k+1}|_{t+2d})\theta_{k+1}^{-N} \leqslant U_t \theta_k^{2d}(1+|u_k|_{t+2d})\theta_{k+1}^{-N} = (U_t\theta_k^{-1})\,(1+|u_k|_{t+2d})\theta_k^{-N};$$

for any fixed t, $\theta_k \geqslant U_t$ for sufficiently large k since θ_k tends to infinity, so that the sequence $(1+|u_k|_{t+2d})\theta_k^{-N}$ is bounded; substituting this into (9), we get an estimate

$$| v_k |_t \leqslant W_t \theta_k^N \tag{12}$$

where $N = 4(2d+1)$ does not depend on t. Now, for any $s \geqslant d$ we can rewrite our interpolation formula with $t = s + (s-d)(N+3)$ and $\bar{\theta}_k = \theta_k^{1/(s-d)}$

$$\begin{aligned} | v_k |_s &\leqslant | S_{\bar{\theta}_k} v_k |_s + | v_k - S_{\bar{\theta}_k} v_k |_s \\ &\leqslant C_{s,d}\bar{\theta}_k^{s-d} | v_k |_d + C_{s,t}\bar{\theta}_k^{s-t} | v_k |_t \\ &\leqslant C_{s,d} V_0 \theta_k^{-3} + C_{s,t} W_t \theta_k^{-3} \end{aligned}$$

where we have used (7) and (12).

Proof of the theorem. Let u_k and v_k be as above. From lemma 2 we have

$$| S_{\theta_j} v_j |_s \leqslant C_{s,s} | v_j |_s \leqslant C_{s,s} V_s \theta_j^{-3}$$

for any $j \geqslant 0$ and $s \geqslant d$, so that the sequence $u_k = u_0 + \sum_{j<k} S_{\theta_j} v_j$ is convergent in every $H^s(\mathbf{R}^n)$ ($\sum_{j \geqslant 0} \theta_j^{-3} < \infty$ by (5)). Moreover, the limit $u \in H^\infty(\mathbf{R}^n)$ of the sequence u_k satisfies

$$\begin{aligned} \| \phi(u) \|_{2d} &\leqslant \| \phi(u_k) \|_{2d} + \left\| \int_0^1 \phi'(u_k + t(u-u_k))\,(u-u_k)dt \right\|_{2d} \\ &\leqslant \| \phi(u_k) \|_{2d} + C_1 | u - u_k |_{3d} \end{aligned}$$

for any k, so that $\phi(u) = 0$ by taking the limit for $k = \infty$.

Application to the local isometric embedding of a Riemannian manifold (following Hörmander [3], Section 2).

Let M be a compact C^∞ manifold of dimension n and g a smooth Riemannian metric on M. In local coordinates, we are thus given a positive definite quadratic form

$$g = \sum_{j,k} g_{jk} dx_j dx_k .$$

The celebrated theorem of Nash [7], which is at the origin of the method, states that for some (large) integer N, there is an isometric embedding $u: M \to \mathbf{R}^N$, that is an injective map satisfying the system of equations

$$\langle \partial_j u, \partial_k u \rangle = g_{jk} \qquad 1 \leqslant j, k \leqslant n \tag{13}$$

where ∂_j stands for $\partial/\partial x_j$ and $\langle \ , \ \rangle$ for the Euclidean scalar product in $\mathbf{R}^N$; thus, any compact Riemannian manifold can be thought as a submanifold of a Euclidean space.

In the proof of this Nash theorem, one first establishes that the set of metrics g such that the problem can be solved is a dense convex cone in the set of all C^∞ metrics on M, and this leads to the following reduced problem (see Hörmander [3] Section 2): show that the equation (13) can be solved for every metric in some neighborhood of a fixed metric g^0.

To illustrate the method described above, let us show how one can use our theorem to prove this last property locally (and this will give a local isometric embedding $u: M \to \mathbf{R}^N$).

Let $\Omega = \{x \in \mathbf{R}^n ; |x| < 1\}$ and choose, near some point $x_0 \in M$, local coordinates such that Ω describes a neighborhood of x_0; we take a $C_0^\infty u_0 : \mathbf{R}^n \to \mathbf{R}^{n(n+3)/2}$ equal to

$$\left((x_j)_{1 \leqslant j \leqslant n}, (x_j^2/2)_{1 \leqslant j \leqslant n}, (x_j x_k)_{1 \leqslant j < k \leqslant n}\right)$$

in a neighborhood of $\bar{\Omega}$; this u_0 is an isometric embedding for the corresponding metric g^0 in Ω, namely the metric $g_{jj}^0 = 1 + |x|^2$ and $g_{jk} = x_j x_k$ if $j \neq k$. Finally, for a metric g close to g^0, we consider the restriction $\phi(u)$ to Ω of the function

$$(\langle \partial_j u, \partial_k u \rangle - g_{jk})_{1 \leqslant j \leqslant k \leqslant n} \tag{14}$$

which is a function in $H^\infty(\Omega)$ valued in $\mathbf{R}^{n(n+1)/2}$ for any $u \in H^\infty(\mathbf{R}^n)$ valued in $\mathbf{R}^{n(n+3)/2}$. Classically, estimates such as (1) hold for $s > (n+2)/2$.

The derivative of ϕ with respect to u is defined by

$$\phi'(u)v = (\langle \partial_j u, \partial_k v \rangle + \langle \partial_k u, \partial_j v \rangle)_{1 \leqslant j \leqslant k \leqslant n} . \tag{15}$$

If $\phi \in H^{\infty}(\Omega)$ is valued in $\mathbf{R}^{n(n+1)/2}$, let us consider it as a function valued in $\mathbf{R}^{n(n+3)/2}$ by adding n components $\varphi_j = 0$ for $1 \leqslant j \leqslant n$, and define $\psi(u)\varphi$ as a continuous extension to $\mathbf{R}^n$ of the function

$$v = -\frac{1}{2}A(u)^{-1}\varphi \tag{16}$$

where $A(u)$ is the $n(n+3)/2$ square matrix the rows of which are $\partial_j u$ for $1 \leqslant j \leqslant n$ and $\partial_j \partial_k u$ for $1 \leqslant j \leqslant k \leqslant n$; thanks to our choice of u_0, the matrix $A(u_0)$ is invertible on Ω, and so is $A(u)$ for any u close enough to u_0. Since $A(u)^{-1}$ is an algebraic function of derivatives of u up to order 2, estimates such as (3) are again classical.

Finally, we have to prove that this operator ψ inverts ϕ' (formula (2)). Applying $A(u)$ to the function v in (16), one gets

$$\langle \partial_j u, v \rangle = -\frac{1}{2}\varphi_j = 0 \qquad 1 \leqslant j \leqslant n$$

$$\langle \partial_j \partial_k u, v \rangle = -\frac{1}{2}\varphi_{jk} \qquad 1 \leqslant j \leqslant k \leqslant n .$$

The x_k derivative of the first equation gives $\langle \partial_j \partial_k u, v \rangle + \langle \partial_j u, \partial_k v \rangle = 0$, and one gets also $\langle \partial_j \partial_k u, v \rangle + \langle \partial_k u, \partial_j v \rangle = 0$ so that the second equation and (15) give $\phi'(u)v = \varphi$ in Ω.

Thus all the assumptions of the theorem are fulfilled, and it follows that we can get a solution if $\phi(u_0)$ is sufficiently small in some $H^s(\Omega)$ norm; but according to (14), $\phi(u_0) = g^0 - g$, and the result is that (13) can be solved for any metric g close enough to g^0, as required.

APPENDIX:

CONSTRUCTION OF THE SMOOTHING OPERATORS IN SOBOLEV SPACES

Let us recall that $v \in H^s(\mathbf{R}^n)$ means $v \in \mathscr{S}'(\mathbf{R}^n)$ and

$$|v|_s^2 = (2\pi)^{-n}\int (1+|\xi|^2)^s\,|\hat{u}(\xi)|^2 d\xi < \infty .$$

Let $\chi: \mathbf{R}^n \to [0, 1]$ be a C^{∞} function taking the value 1 in a neighborhood of 0 and vanishing for $|\xi| \geqslant \sqrt{3}$. For $v \in H^{\infty}(\mathbf{R}^n)$ and $\theta > 1$ one sets

$$\widehat{S_\theta v}(\xi) = \chi(\xi/\theta)\hat{v}(\xi) .$$

Then, if $s \geqslant t$,

$$\begin{aligned}(1+|\xi|^2)^s\,|\widehat{S_\theta v}(\xi)|^2 &\leqslant \theta^{2(s-t)}(1+|\xi/\theta|^2)^{s-t}\,|\chi(\xi/\theta)|^2(1+|\xi|^2)^t\,|\hat{v}(\xi)|^2 \\ &\leqslant (2\theta)^{2(s-t)}(1+|\xi|^2)^t\,|\hat{v}(\xi)|^2\end{aligned}$$

since $|\chi| \leqslant 1$ and $|\xi/\theta| \leqslant \sqrt{3}$ for $(\xi/\theta) \in \operatorname{supp} \chi$; this gives the first estimate (4) with $C_{s,t} = 2^{s-t}$.

Similarly, for $s \leqslant t$,

$$(1+|\xi|^2)^s \,|\, \hat{v}(\xi) - \widehat{S_\theta v}(\xi) \,|^2 = |\, 1 - \chi(\xi/\theta) \,|^2 (1+|\xi|^2)^s \,|\, \hat{v}(\xi) \,|^2 \,;$$

a Taylor formula gives $|\, 1 - \chi(\xi/\theta) \,| \leqslant C_k \,|\, \xi/\theta \,|^k$ with $C_k = \sup |\, \chi^{(k)} \,| / k!$ for any $k \in \mathbf{N}$ since $\chi(0) = 1$ and $\chi^{(j)}(0) = 0$ for $j > 0$, so that for $t = s + k$

$$\begin{aligned}(1+|\xi|^2)^s \,|\, \hat{v}(\xi) - \widehat{S_\theta v}(\xi) \,|^2 &\leqslant C_{t-s}^2 \,|\, \xi/\theta \,|^{2(t-s)} (1+|\xi|^2)^s \,|\, \hat{v}(\xi) \,|^2 \\ &\leqslant C_{t-s}^2 \theta^{2(s-t)} (1+|\xi|^2)^t \,|\, \hat{v}(\xi) \,|^2\end{aligned}$$

whence the second estimate (4) with $C_{s,t} = C_{t-s} = \sup |\, \chi^{(t-s)} \,| / (t-s)!$

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