4. Relative de Rham homology

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **35 (1989)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **04.06.2024**

If we combine theorem (3.2) and the biduality theorem (2.1) we obtain what is usually known as the

(3.6) DE RHAM THEOREM. *Integration over smooth singular simplexes induces an isomorphism*

$$H^{\bullet}(X, \mathbf{C}) \xrightarrow{\sim} H^{\bullet}_{\infty}(X, \mathbf{C})$$

from de Rham cohomology to smooth singular cohomology.

4. RELATIVE DE RHAM HOMOLOGY

Let us start by some general remarks on the support of a compact p-chain T on a smooth n-dimensional manifold X. Since we can realize T as a section in the sheaf $\Omega^{p\vee}$ the general sheaf theoretic notion of support applies: The *support* of T, $\mathrm{Supp}(T)$ is the smallest closed subset Z of X, such that the restriction of T to $X - Z$ is zero.

(4.1) EXAMPLE. Integration over an oriented compact p-dimensional submanifold K with boundary defines a compact p-chain κ with $\mathrm{Supp}(\kappa) = K$. From Stokes formula

$$\int_K d\omega = \int_{\partial K} \omega\,, \quad \omega \in \Gamma(X, \Omega^p)\,,$$

we conclude that $\mathrm{Supp}(b\kappa) = \partial K$.

Let us now consider the inclusion $j\colon U \to X$ of an open subset U of X. The induced map

$$j_*\colon D^c_p(U, \mathbf{C}) \to D^c_p(X, \mathbf{C})\,, \quad p \in \mathbf{N}\,,$$

is injective since we may interpret j_* as "extension by zero" in the sheaf $\Omega_p^{\vee}$, compare (2.5). A compact p-chain T on X belongs to the image of j_* if and only if $\mathrm{Supp}(T) \subseteq U$. The complex $D^c_{\bullet}(X, U; \mathbf{C})$ of *relative compact chains* is defined to fit into the following exact sequence

(4.2) $$0 \to D^c_{\bullet}(U, \mathbf{C}) \xrightarrow{j_*} D^c_{\bullet}(X, \mathbf{C}) \to D^c_{\bullet}(X, U; \mathbf{C}) \to 0\,.$$

On this basis we can define the *relative de Rham* homology group

$$H_p(X, U; \mathbf{C}) = H_p D^c_{\bullet}(X, U; \mathbf{C})\,, \quad p \in \mathbf{N}\,.$$

In more concrete terms we can describe this homology group as

$$(4.3)\qquad \{Z \in D_p^c(X, \mathbf{C}) \mid \operatorname{Supp}(bZ) \subseteq U\} \Big/ \begin{array}{l}\{bW \mid W \in D_{p+1}^c(X, \mathbf{C})\} \\ \quad + \{Z \in D_p(X, \mathbf{C}) \mid \operatorname{Supp}(Z) \subseteq U\}\end{array}$$

From the exact sequence (4.2) we deduce the homology sequence

$$(4.4)\qquad \begin{array}{c} \to H_p^c(U, \mathbf{C}) \to H_p^c(X, \mathbf{C}) \to H_p^c(X, U; \mathbf{C}) \\ \to H_{p-1}^c(U, \mathbf{C}) \to H_{p-1}^c(X, \mathbf{C}) \to \end{array}$$

Let $f: X \to Y$ denote a smooth map, U an open subset of X and V an open subset of Y containing $f(U)$. Let us notice that

$$(4.5)\qquad \operatorname{Supp}(f_* T) \subseteq f(\operatorname{Supp}(T)), \quad T \in D_p^c(X, \mathbf{C}).$$

These remarks make it evident, that de Rham homology is a covariant functor on the category of pairs consisting of a manifold and one of its open subspaces.

(4.6) *Excision.* Let Z be a closed subset of X and Y an open subset of X containing Z. The inclusion of $V = Y - Z$ in $U = X - Z$ induces an isomorphism

$$H_\bullet^c(Y, V; \mathbf{C}) \tilde{\to} H_\bullet^c(X, U; \mathbf{C}).$$

Proof. Let $i: Z \to X$ denote the inclusion. From the fact that $\Omega^{\bullet\vee}$ consists of soft sheaves we deduce an exact sequence

$$0 \to \Gamma_c(U, \Omega^{\bullet\vee}) \to \Gamma_c(X, \Omega^{\bullet\vee}) \to \Gamma_c(Z, i^*\Omega^{\bullet\vee}) \to 0$$

compare [5] III. 7.6. This alows us to make the identification

$$(4.7)\qquad D_\bullet^c(X, U; \mathbf{C}) \tilde{\to} \Gamma_c(Z, i^*\Omega^{\bullet\vee}), \quad Z = X - U.$$

The expression on the right hand side is unchanged, when X is replaced by Y and U by V. Q.E.D.

(4.8) *Continuity.* Let (X_α) be an outward directed open covering of the manifold X. For any open subset U of X we have that

$$\lim_{\to\alpha} H_\bullet^c(X_\alpha, U \cap X_\alpha; \mathbf{C}) = H_\bullet^c(X, U; \mathbf{C})$$

Proof. As a consequence of the theorem of Borel-Heine, see possibly [5] III. 5.2, we find that

$$\lim_{\to} D_\bullet^c(X_\alpha, \mathbf{C}) = D_\bullet^c(X, \mathbf{C})$$

and similarly with X replaced by U and X_α replaced by $U \cap X_\alpha$. Using this and the exact sequence 4.2 we get that

$$\varinjlim D^c_\bullet(X_\alpha, U \cap X_\alpha; \mathbf{C}) = D^c_\bullet(X, U; \mathbf{C})$$

from which the result follows by passing to homology. Q.E.D.

Let us also notice that in case X is the disjoint union of a family (X_α) of open subsets we have that

$$\bigoplus_\alpha H^c_\bullet(X_\alpha, U \cap X_\alpha; \mathbf{C}) \xrightarrow{\sim} H^c_\bullet(X, U; \mathbf{C}) . \tag{4.9}$$

5. Stokes formula

Let us consider the open subset U of the n-dimensional smooth manifold X and the resulting exact sequences

$$(5.1)\quad \begin{gathered} \to H^c_p(X, \mathbf{C}) \to H^c_p(X, U; \mathbf{C}) \xrightarrow{b} H^c_{p-1}(U, \mathbf{C}) \xrightarrow{j_*} H^c_{p-1}(X, \mathbf{C}) \to \\ \leftarrow H^p(X, \mathbf{C}) \leftarrow H^p(X, U; \mathbf{C}) \xleftarrow{\partial} H^{p-1}(U, \mathbf{C}) \xleftarrow{j^*} H^{p-1}(X, \mathbf{C}) \leftarrow \end{gathered}$$

where the first is discussed in the previous section and the second is the sheaf cohomology sequence. The relative term in the second sequence is often written

$$H^p_Z(X, \mathbf{C}) = H^p(X, U; \mathbf{C}) , \quad Z = X - U . \tag{5.2}$$

We can now extend the biduality theorem (2.1).

(5.3) Theorem. *The cohomology sequence above is dual to the homology sequence. In particular we have a Stoke's formula*

$$< b\alpha, \omega > = < \alpha, \partial\omega >$$

for $\alpha \in H^c_p(X, U; \mathbf{C})$ *and* $\omega \in H^{p-1}(U, \mathbf{C})$.

Proof. The first sequence arises from the following short exact sequence of complexes, compare (4.2) and (4.7),

$$0 \to \Gamma_c(U, \Omega^{\bullet\vee}) \xrightarrow{j_*} \Gamma_c(X, \Omega^{\bullet\vee}) \to \Gamma_c(Z, \Omega^{\bullet\vee}) \to 0 .$$

In order to calculate the second sequence we depart from the flabby resolution $\Omega^{\bullet\vee\vee}$ of $\mathbf{R}$ established in the proof of the biduality theorem (2.1).