2. The composition product of homfly polynomials

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the homfly polynomial has been found, apart from the case of the Alexander-Conway polynomial. The purpose of this paper is to present some progress towards the solution of these problems.

In Section 2 we introduce a composition product for homfly polynomials. This product allows the combinatorial definition of the homfly polynomial of a diagram for a given pair of values of the variables in terms of the homfly polynomials of its subdiagrams for other related pairs of values of the variables (Proposition 1). We show in Proposition 2 how the sequence of state models due to Jones can be derived simply from the product operation, starting from an elementary special case of the homfly polynomial. Then, motivated by some difficulties in the application of the concept of composition product to the Alexander-Conway polynomial, in Section 3 we restrict our attention to closed braids and we introduce a specified composition product for this class of diagrams (Proposition 3). This leads us first to another version of the Jones sequence of state models (Proposition 4). Then we obtain a state model for the Alexander-Conway polynomial (Proposition 6) which can be interpreted as an ice-type model (Proposition 7). As another consequence we give an expansion of the homfly polynomial of a braid diagram in terms of the Alexander-Conway polynomials of its subdiagrams (Proposition 9). This yields simple direct proofs of some inequalities due to Morton [22] and independently Franks and Williams [4] which have been helpful in the study of the braid index. Finally we combine the previous results to obtain a state model for the homfly polynomial of a closed braid (Proposition 12). We present some perspectives for further research in Section 4.

2. The composition product of homfly polynomials

2.1. Definitions

By *diagram* we mean a regular plane projection of a tame oriented link in 3-space. We shall consider diagrams as 4-regular directed plane graphs.

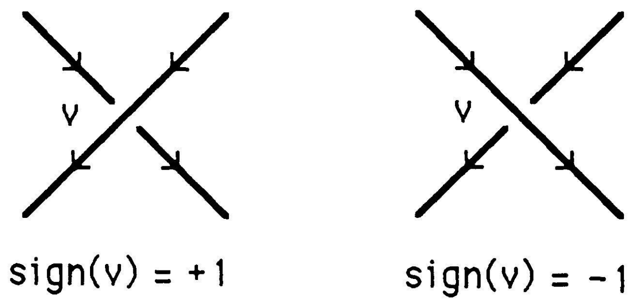


Figure 1

In particular a simple Jordan curve (corresponding to the trivial knot) is a graph with no vertices and one edge which we call the *free loop*. The vertices of a diagram will be signed according to the convention described in Figure 1.

The *writhe* of the diagram D, denoted by $w(D)$, is the sum of the signs of the vertices of D. We define the *rotation number* of D, denoted by $r(D)$, as the sum of the signs of the Seifert circles of D, where the sign of such a circle is 1 if it is oriented counterclockwise and -1 otherwise (this combinatorial form of the Whitney degree appears in [16] p. 95-100, where it is called *curliness*).

Two diagrams will be said to be *isotopic* if they represent the same oriented link up to ambient isotopy. We shall need the following economical form of Reidemeister's Theorem given in [28]: two diagrams are isotopic if and only if one can be obtained from the other by a finite sequence of moves of types A_i, A_i', B_i, B_i' $(i=1, 2, 3, 4)$ and C, C' described in Figure 2.

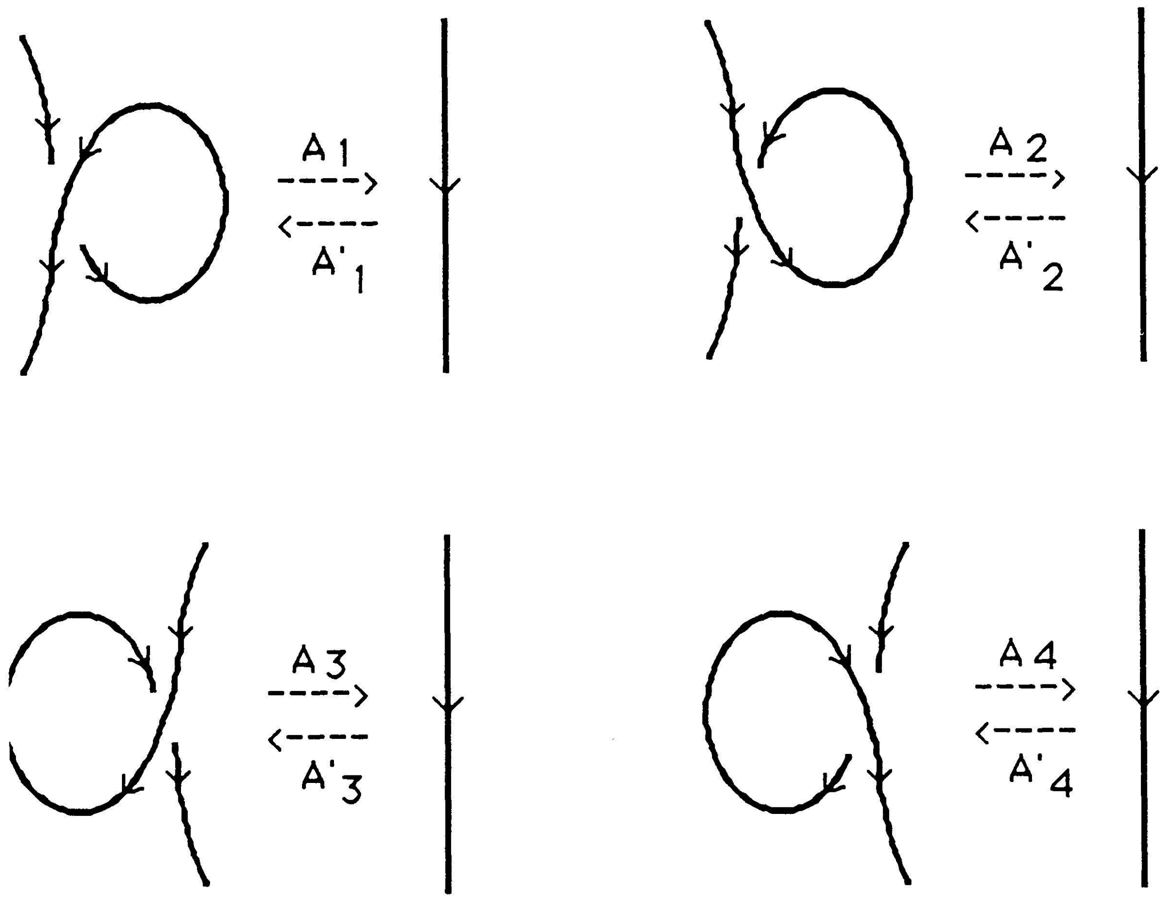


FIGURE 2A

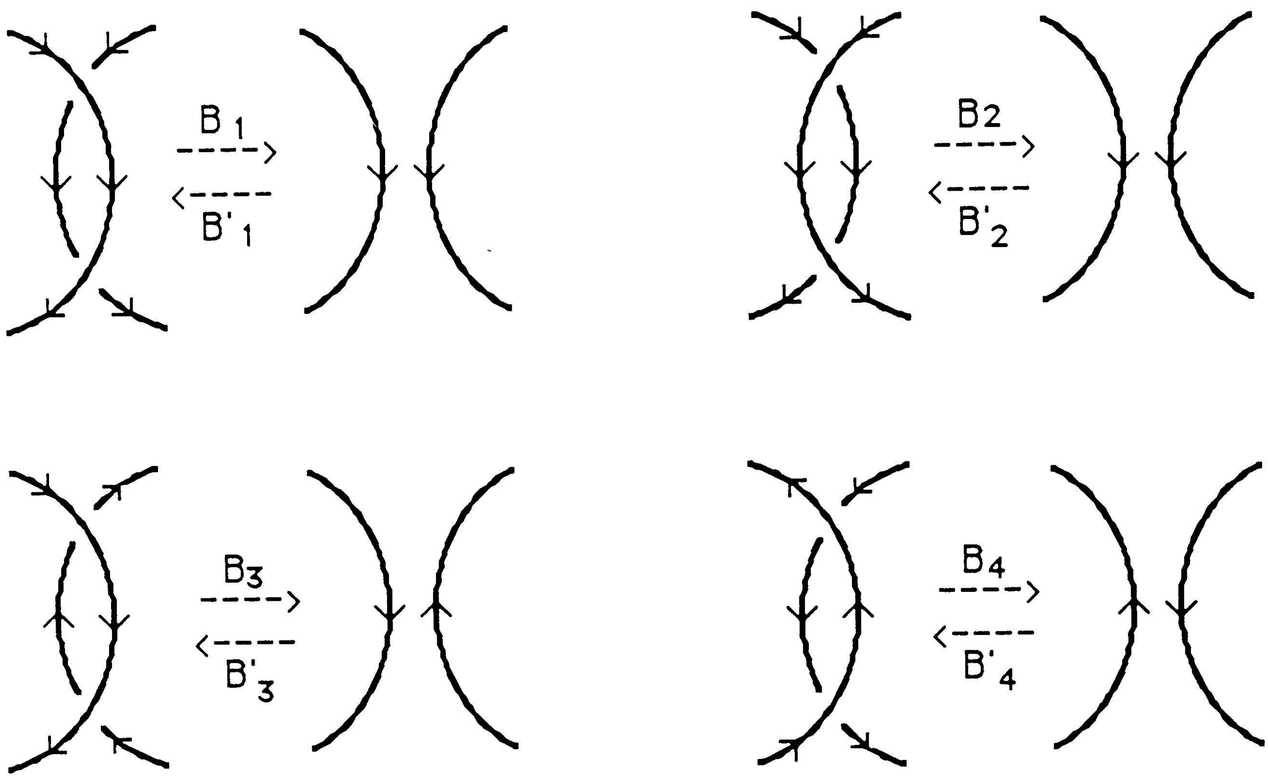


FIGURE 2B

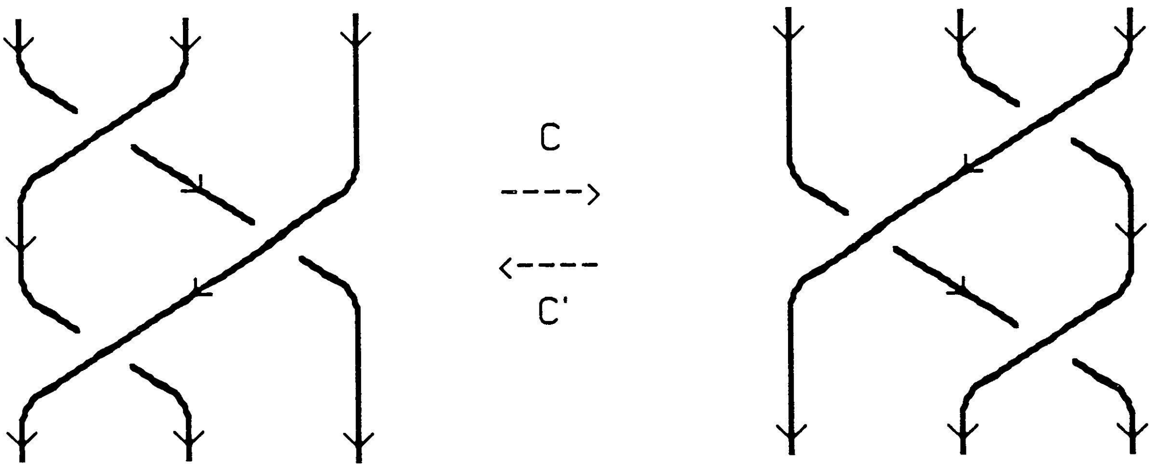


FIGURE 2C

Two diagrams will be said to be *regularly isotopic* if one can be obtained from the other by a finite sequence of moves of types $B_i, B'_i\,(i=1,2,3,4)$ and C, C' (this concept is due to Kauffman [19]). The writhe and the rotation number are invariants of regular isotopy: if D and D' are regularly isotopic diagrams, then $w(D) = w(D')$ (this is immediate) and $r(D) = r(D')$ (see [16] p. 95-100).

If D^+, D^- and D° are diagrams which are identical outside a small disk and behave as depicted in Figure 3 inside that disk, we shall say that (D^+, D^-, D°) form a *Conway triple*.

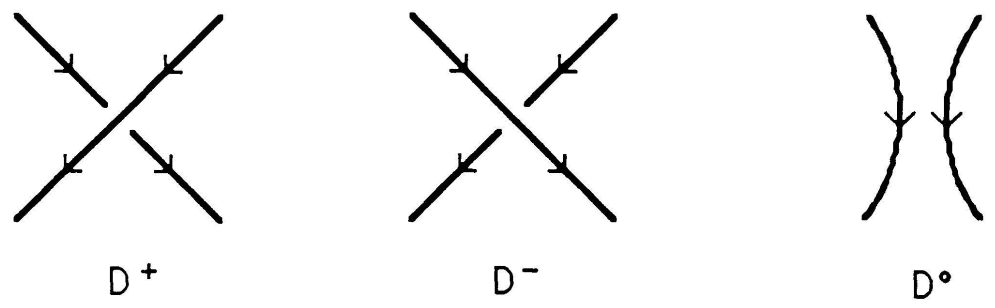


FIGURE 3

We shall be concerned here with the following result:

THEOREM. *One can associate (in an unique way) to every diagram D a Laurent polynomial with integer coefficients in two variables z, a which we denote by $H(D, z, a)$ in such a way that the following properties hold:*

(i) *If D and D' are regularly isotopic, $H(D, z, a) = H(D', z, a)$.*

(ii) *If D' is obtained from D by a move of type A'_1 or A'_3 (respectively: A'_2 or A'_4) then $H(D', z, a) = aH(D, z, a)$ (respectively: $H(D', z, a) = a^{-1}H(D, z, a)$).*

(iii) *If (D^+, D^-, D°) form a Conway triple then:*

$$H(D^+, z, a) - H(D^-, z, a) = zH(D^\circ, z, a).$$

(iv) *If D is the free loop, $H(D, z, a) = 1$.*

Then, as observed in [19], if we set $P(D, z, a) = a^{-w(D)}H(D, z, a)$ this defines the following version of the homfly polynomial [5, 6, 8, 21, 25, 26]: P is an isotopy invariant which takes the value 1 on the free loop and satisfies, for every Conway triple (D^+, D^-, D°): $aP(D^+, z, a) - a^{-1}P(D^-, z, a) = zP(D^\circ, z, a)$. In particular, $P(D, z, 1) = H(D, z, 1)$ is the Alexander-Conway polynomial of D [1, 3, 15, 16] and we denote it by $A(D, z)$.

We shall need the following easy consequence of the above Theorem:

(v) If the diagram D' is obtained from the diagram D by the addition of a single free loop, then $H(D', z, a) = (a-a^{-1})z^{-1}H(D, z, a)$.

For the sake of simplicity we define: $H'(D, z, a) = (a-a^{-1})z^{-1}H(D, z, a)$. Thus H' can replace H in properties (i), (ii), (iii) of the Theorem, and satisfies

(iv′) If D is the free loop, $H'(D, z, a) = (a-a^{-1})z^{-1}$.

In the sequel we shall have to consider the *empty diagram* ω, which has no vertices and no edges. It will be convenient to set $H'(\omega, z, a) = 1$, so that property (v) is valid with H' instead of H even when D is the empty diagram. This convention together with property (v) can replace property (iv′) in the definition of H'.

2.2. Labellings and the composition product

We define a *labelling* of a diagram D as a mapping f from the edge-set of D to the set of positive integers which satisfies the following

conservation law: for every positive integer i, at every vertex v of D, the number of edges labelled i (that is, edges in $f^{-1}(i)$) incident towards v equals the number of such edges incident from v (a loop at v contributing 1 to both numbers).

Then if we first erase all edges not labelled i and the isolated vertices thus created, "smoothing out" all vertices of degree 2 (see Figure 4) and retaining the signs (or equivalently the crossing structure) at every vertex of degree 4, we obtain a (possibly empty) diagram which we denote by $D_{f,i}$ and call a *subdiagram* of D. We may associate to every edge e of D with $f(e) = i$ a unique simple (possibly closed) directed path in D containing e which is converted by the above process into an edge of $D_{f,i}$. This edge of $D_{f,i}$ will be denoted by $P_f(e)$. Thus we have defined a mapping P_f from the edge-set of D to the union of the edge-sets of all $D_{f,i}$. We call this mapping P_f the *projection associated to* f.

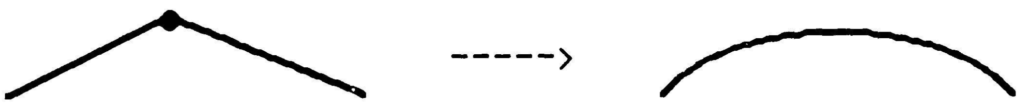


FIGURE 4

For any labelling f of the diagram D, we may write $r(D) = \sum_i r(D_{f,i})$ (with the obvious convention that an empty diagram contributes zero to this sum). This additivity property of the rotation number is immediate from the definition of this number as a Whitney degree (see [16] p. 95-100) and we shall use it implicitly in the sequel.

We define the *interaction* $<v \mid D \mid f>$ of the vertex v in the diagram D with the labelling f as follows. If the edges incident to v are assigned only one label, or two distinct labels i and j in such a way that $D_{f,i}$ and $D_{f,j}$ cross at v, then $<v \mid D \mid f> = 1$. Otherwise $<v \mid D \mid f>$ is defined on Figure 5. If W is some set of vertices of D we write $<W \mid D \mid f> = \prod_{v \in W} <v \mid D \mid f>$. We shall take $<W \mid D \mid f>$ as equal to 1 if W is empty. We write more briefly $<D \mid f>$ for $<V \mid D \mid f>$ if D has vertex-set V.

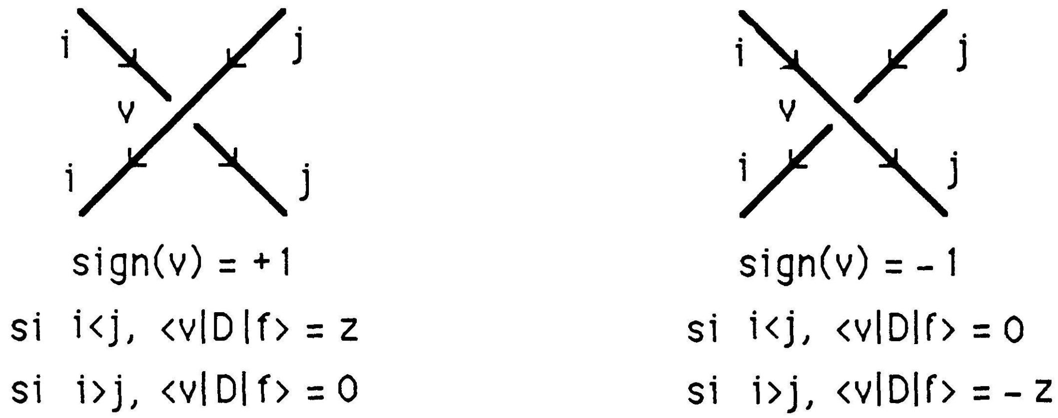


FIGURE 5

We denote by $L(D, k)$ the set of labellings of D which take their values in $\{1, \dots, k\}$.

PROPOSITION 1. *For any diagram* D,

$$\sum_{f \in L(D,2)} \langle D \mid f \rangle a_2^{-r(D_{f,1})} a_1^{r(D_{f,2})} H'(D_{f,1}, z, a_1) H'(D_{f,2}, z, a_2) = H'(D, z, a_1 a_2).$$

Proof. Let us write $H''(D, z, a_1, a_2) = \sum_{f \in L(D,2)} C(D, f)$ with

$$C(D, f) = \langle D \mid f \rangle a_2^{-r(D_{f,1})} a_1^{r(D_{f,2})} H'(D_{f,1}, z, a_1) H'(D_{f,2}, z, a_2).$$

We shall show that the expression $H''(D, z, a_1, a_2)$ satisfies properties (i), (ii), (iii), (v) of Section 2.1 with $a = a_1 a_2$. We now introduce the general method which will be used in the different cases.

Consider a pair of diagrams (D, D') which are identical outside some disk and take specified forms inside this disk. This is the case when D and D' are related by a Reidemeister move (Figure 2 specifies the forms inside the disk). Similarly if D' is obtained from D by the addition of a free loop with empty interior, we may consider a disk which is empty in D and contains this free loop in D', so that D and D' are identical outside that disk. We shall call such a disk a *separator* and D and D' will be said to be *compatible* with respect to this separator. Similarly a disk involved in the definition of a Conway triple will also be called a separator with respect to which the elements of the triple are compatible.

Consider now a diagram D with a given separator S. Vertices of D situated in the interior (respectively: exterior) of S will be called *inner* (respectively: *outer*), and we assume that there are no other vertices. An edge of D which meets the exterior of S will also be called *outer*. If we shrink S into a new vertex s we obtain a plane graph which we call the *outerdiagram* associated to (D, S). An edge of the outerdiagram which is incident to s will be called a *boundary edge*. Such an edge e can be identified with a portion of some outer edge of D which crosses the boundary of S and we shall denote this unique outer edge by $o(e)$. Similarly an edge e of the outerdiagram which is not a boundary edge can be identified with a unique outer edge of D also denoted by $o(e)$.

We call *outer labelling* of D (with respect to S) a mapping from the set of edges of the associated outerdiagram to the set of positive integers which satisfies the conservation law at every outer vertex. Then clearly the conservation law also holds at the special vertex s. We denote the set of outer labellings of D with values in $\{1, 2\}$ by $L^{\circ}(D, 2)$. For f in $L^{\circ}(D, 2)$ and g in $L(D, 2)$ we write $f \subseteq g$ to indicate that f can be obtained from g by "labelled shrinking", in other words that for every edge e of the outerdiagram, $f(e) = g(o(e))$.

Now we may write: $H''(D, z, a_1, a_2) = \sum_{f \in L^{\circ}(D, 2)} C(D, f)$, with

$$C(D, f) = \sum_{g \in L(D, 2),\, f \subseteq g} C(D, g)\,.$$

The properties (i), (ii), (iii), (v) to be proved take the following form:

$$\sum_i x_i H''(D_i, z, a_1, a_2) = 0$$

for some family of diagrams (a pair or a triple) (D_i) compatible with respect to a separator S. Thus all diagrams of this family have the same set of outer labellings with values in $\{1, 2\}$.

We shall show that for every such outer labelling f: $\sum_i x_i C(D_i, f) = 0$.

For this purpose we introduce a *reference* consisting of a diagram R compatible with the D_i (with respect to S) together with a labelling h in $L(R, 2)$ with $f \subseteq h$. Then evaluating $C = \left(\sum_i x_i C(D_i, f)\right)/C(R, h)$ instead of $\sum_i x_i C(D_i, f)$ will yield substantial simplification.

To be more precise, recall that for every diagram D in the family $((D_i), R)$ and every $g \in L(D, 2)$ with $f \subseteq g$:

$$C(D, g) = \langle D \mid g \rangle a_2^{-r(D_{g,1})} a_1^{r(D_{g,2})} H'(D_{g,1}, z, a_1) H'(D_{g,2}, z, a_2)$$

Then, denoting by $V^{\circ}(D)$ the set of outer vertices and by $V^{i}(D)$ the set of inner vertices of D, we have $\langle D \mid g \rangle = \langle V^{\circ}(D) \mid D \mid g \rangle \; \langle V^{i}(D) \mid D \mid g \rangle$. Clearly $\langle V^{\circ}(D) \mid D \mid g \rangle$ does not depend on the choice of D in $((D_i), R)$ and g in $L(D, 2)$ with $f \subseteq g$, and we may denote it by $\langle f \rangle$.

If $\langle f \rangle = 0$ clearly $\sum_i x_i C(D_i, f) = 0$ and we are done. In the sequel we consider only outer labellings f such that $\langle f \rangle \neq 0$. Then in evaluating C

we shall divide all contributions by $<f>$, which amounts to the replacement of the interaction $<D \mid g>$ by the *inner interaction* $<V^i(D) \mid D \mid g>$. We shall always choose R and h in such a way that $<V^i(R) \mid R \mid h> \neq 0$.

Then for every diagram D in the family (D_i) and every $g \in L(D, 2)$ with $f \subseteq g$ we shall write $C'(D, g) = C(D, g)/C(R, h)$ as an ordered product $T_1 . T_2 . T_3 . T_4 . T_5$ with

$$T_1 = <V^i(D) \mid D \mid g>/<V^i(R) \mid R \mid h>\,; \; T_2 = a_2^{-r(D_{g,1})+r(R_{h,1})};$$
$$T_3 = a_1^{r(D_{g,2})-r(R_{h,2})}; \; T_4 = H'(D_{g,1}, z, a_1)/H'(R_{h,1}, z, a_1);$$
$$T_5 = H'(D_{g,2}, z, a_2)/H'(R_{h,2}, z, a_2)\,.$$

If $T_1 = 0$ the other terms will not be evaluated. We shall denote by $C'(D, f)$ the sum

$$\textstyle\sum_{g \in L(D,2),\, f \subseteq g} C'(D, g) = C(D, f)/C(R, h)\,.$$

Proof of property (v). Let the diagram D' be obtained from the diagram D by the addition of a single free loop O. In order to use the geometric concept of separator as defined above, we assume that the interior of O is empty (otherwise the proof would be essentially the same. Let $r(O) = \varepsilon \in \{+1, -1\}$. The outerdiagram is D (together with a new isolated vertex s) and we consider the given outer labelling f as a labelling of D. We take as a reference $R = D$ and $h = f$.

We must show that $C'(D', f) = (a_1 a_2 - (a_1 a_2)^{-1}) z^{-1}$.

This is done on Figure 6 which displays the contributions

$$C'(D', g)\,(g \in L(D', 2),\, f \subseteq g)\,,$$

written as ordered products $T_1 . T_2 . T_3 . T_4 . T_5$ as specified above and evaluated using property (v) for H'

$1.a_2^{-\varepsilon}.1.(a_1 - a_1^{-1})z^{-1}.1$ $\qquad$ $1.1.a_1^{\varepsilon}.1.\,(a_2 - a_2^{-1})z^{-1}$

FIGURE 6

Note that the above proof also works when D is empty, so that property (iv′) indeed holds for H'' with $a = a_1 a_2$.

Proof of property (iii). Let (D^+, D^-, D°) be a Conway triple and f be a given outer labelling.

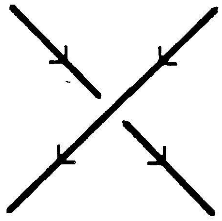
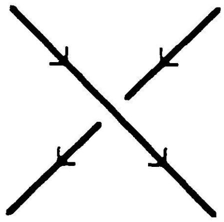
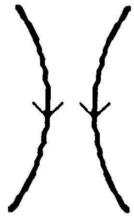
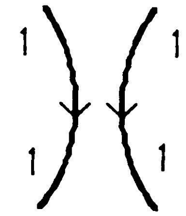
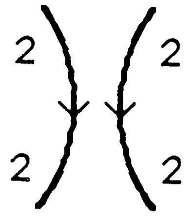
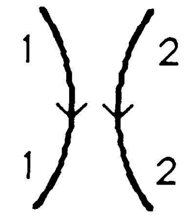
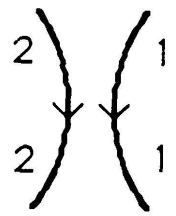
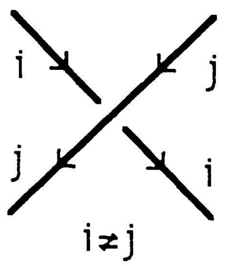
			
Reference	D^+	D^-	D°
	$1.1.1.X^+.1$	$1.1.1.X^-.1$	1
	$1.1.1.1.Y^+$	$1.1.1.1.Y^-$	1
	z.1.1.1.1.	0	1
	0	-z.1.1.1.1	1
	1.1.1.1.1	1.1.1.1.1	0

FIGURE 7

We must show that $C'(D^+, f) - C'(D^-, f) = zC'(D^\circ, f)$. The proof is given on Figure 7 which lists the various contributions. There are six cases to consider according to the labels of the boundary edges. These labels are shown on the picture representing the part of the reference diagram situated inside the separator. The labellings of all diagrams are determined uniquely by the outer labellings and are not described. In the first case, the reference is D° and X^+ denotes $H'(D^+_{g,1}, z, a_1)/H'(D^\circ_{h,1}, z, a_1)$ for the unique element g of $L(D^+, 2)$ such that $f \subseteq g$. X^- is defined similarly and the equality $X^+ - X^- = z$ follows from property (iii) for H'. The second case is settled in exactly the same way. The third and the fourth case are immediate. For the remaining two cases we note that $C'(D^\circ, f) = 0$ because there is no element g of $L(D^\circ, 2)$ such that $f \subseteq g$.

Proof of property (ii). We first observe that it is enough to consider moves of type A_1 and A_3. The move of type A_2 can be reduced to the move of type A_1, as proved diagrammatically on Figure 8, using properties (iii), (v) which have already been established. Here as usual we depict only the portions of diagrams where modifications occur, and each diagram D stands for $H''(D, z, a_1, a_2)$ (we write $a = a_1 a_2$). The proof of the reduction of the move of type A_4 to the move of type A_3 will be obtained by reversing all arrows on Figure 8.

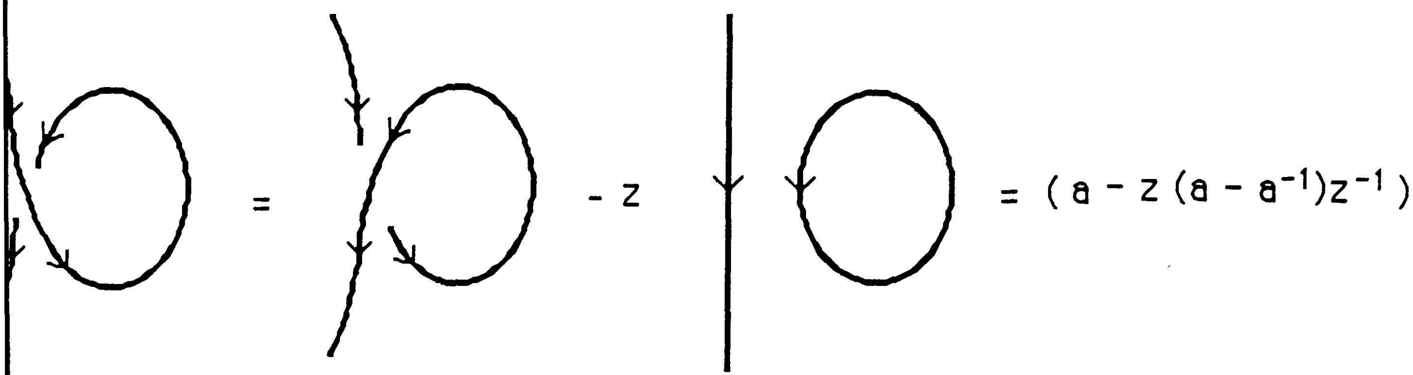


FIGURE 8

Now let D' be obtained from D by a move of type A'_1 or A'_3 (see Figure 2A), and consider an outer labelling f. We take as a reference $R = D$ and the unique element h of $L(D, 2)$ such that $f \subseteq h$. We must show that $C'(D', f) = a_1 a_2$.

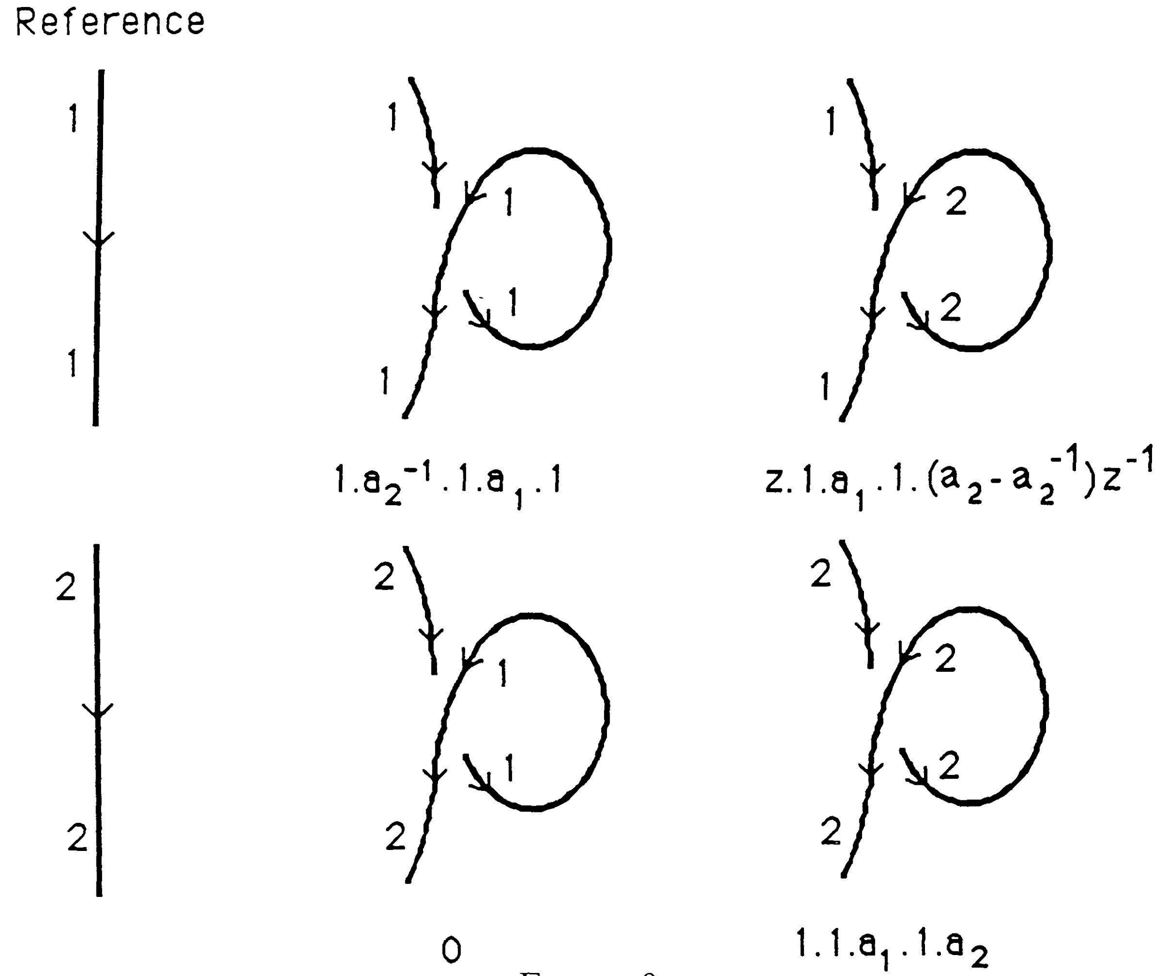


FIGURE 9

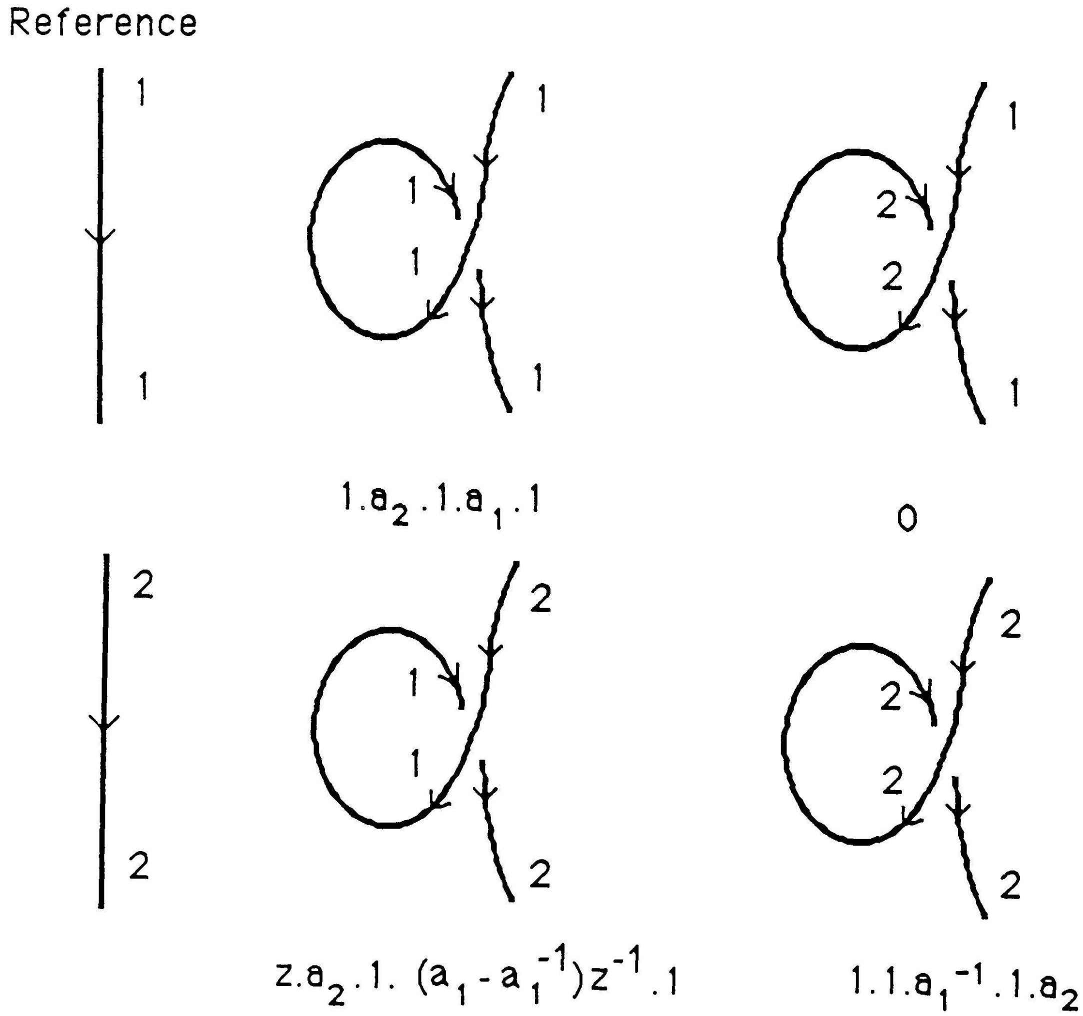


FIGURE 10

The proof is described on Figures 9 (move of type A'_1) and 10 (move of type A'_3) with the same conventions as above. In each case properties (ii) and (v) of H' are used.

Proof of property (i). First we show on Figure 11 that the move of type B_2 can be reduced, by using the already established property (iii), to the move of type B_1. As before, in this figure each diagram D stands for $H''(D, z, a_1, a_2)$.

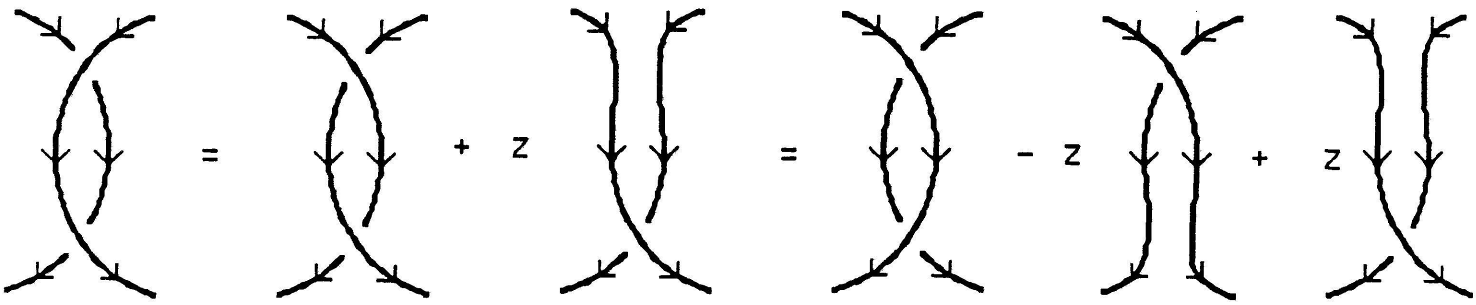


FIGURE 11

Let D' be obtained from D by a move of type B'_1, B'_3 or B'_4 (see Figure 2B), whose effect is studied in Figures 12, 13, 14 respectively. Let f be a given outer labelling. For each move there are six cases to consider according to the labels of the boundary edges. We take as a reference $R = D$ and the unique element h of $L(D, 2)$ such that $f \subseteq h$ whenever such an h exists (this corresponds to the first four cases). Then, using property (i) for H' in the two first cases, it is easy to check directly on Figures 12, 13, 14 that $C'(D', f) = 1$.

In the remaining two cases there is no labelling g in $L(D, 2)$ such that $f \subseteq g$. We choose a suitable reference and, using property (ii) for H' in Figures 13 and 14, we check that $C'(D', f) = 0$.

Finally let D' be obtained from D by a move of type C (see Figure 2C), and consider an outer labelling f. We first classify the different cases for f according to the labellings of the boundary edges which are oriented from the exterior of the separator towards its interior. For each case we describe for both diagrams D and D' all labellings of the edges incident to the inner vertices which will yield a non-zero inner interaction. This is done in Figures 15 to 21. In each figure one part corresponds to D and the other to D', and each labelling appears with its inner interaction.

We must show that, after the choice of a suitable reference, for every outer labelling f, $C'(D, f) = C'(D', f)$.

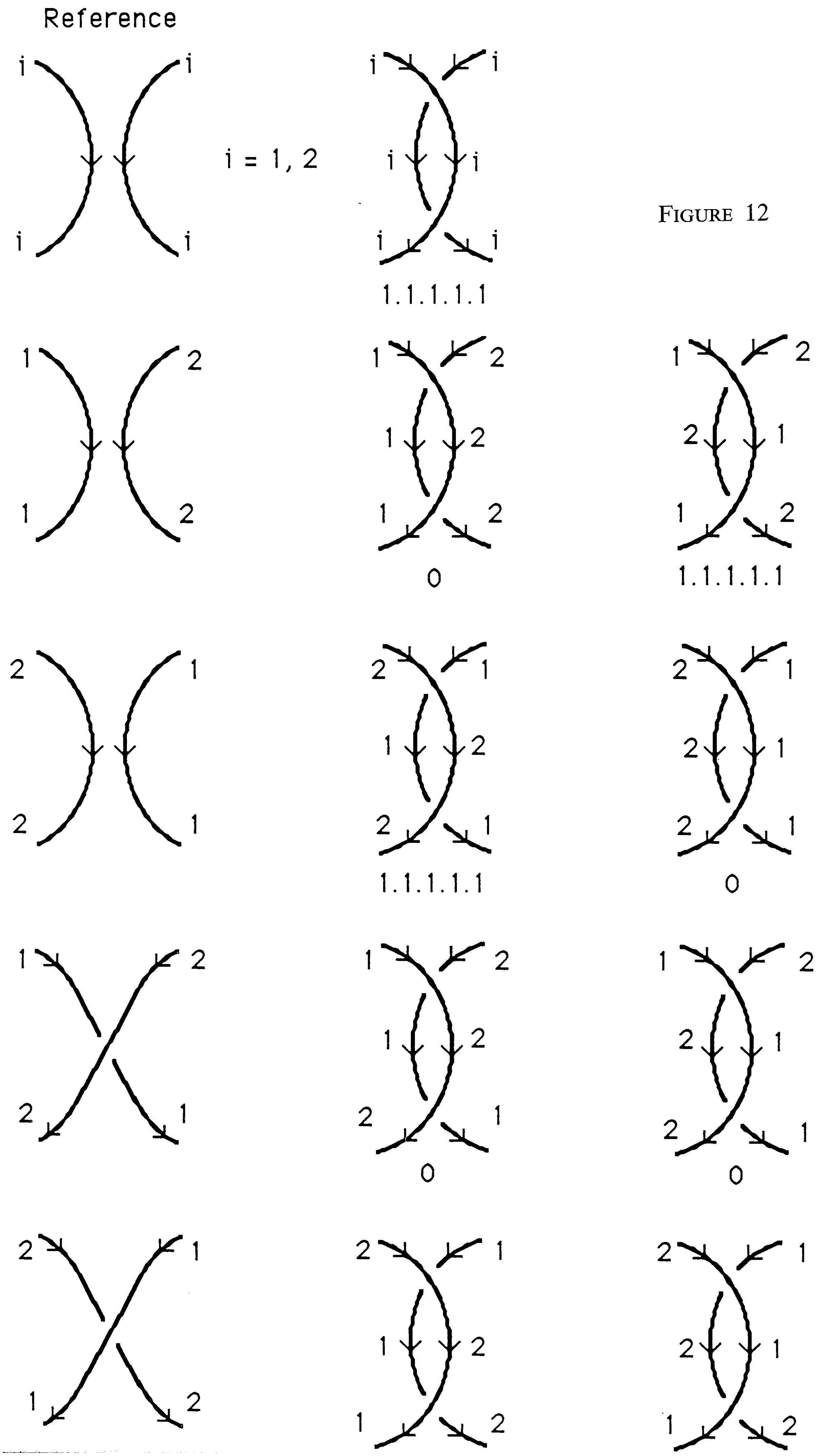


FIGURE 12

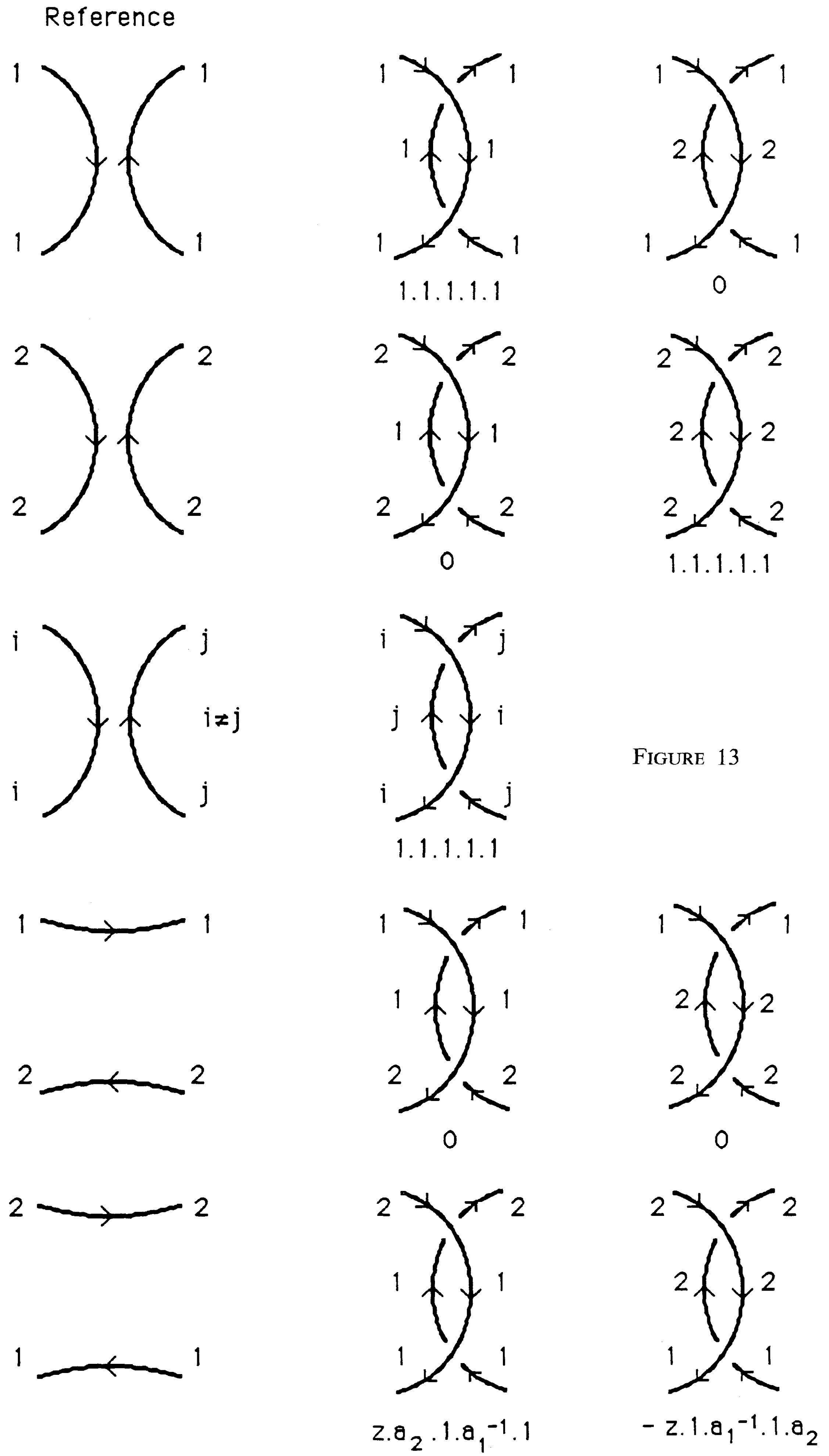


FIGURE 13

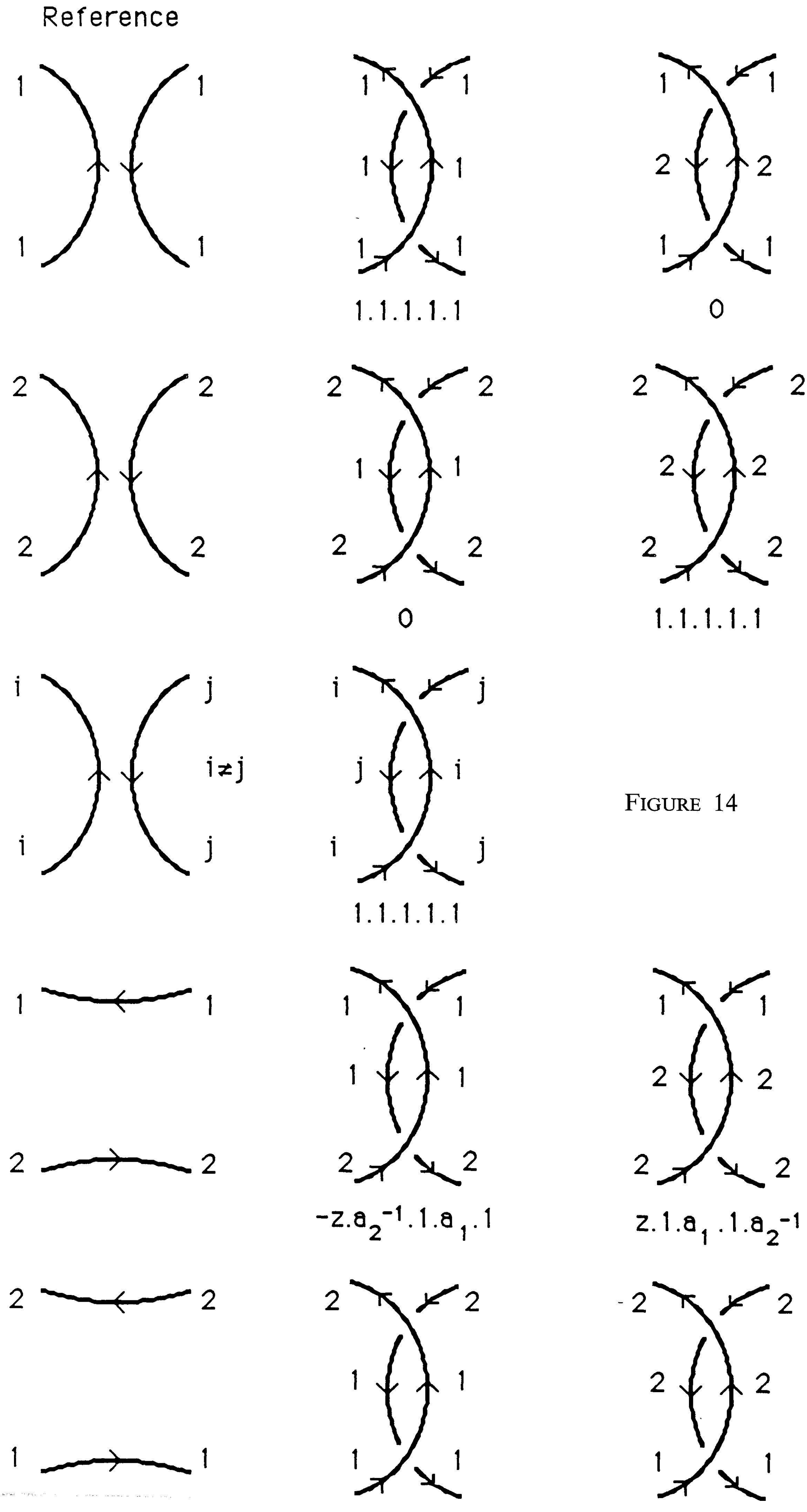


FIGURE 14

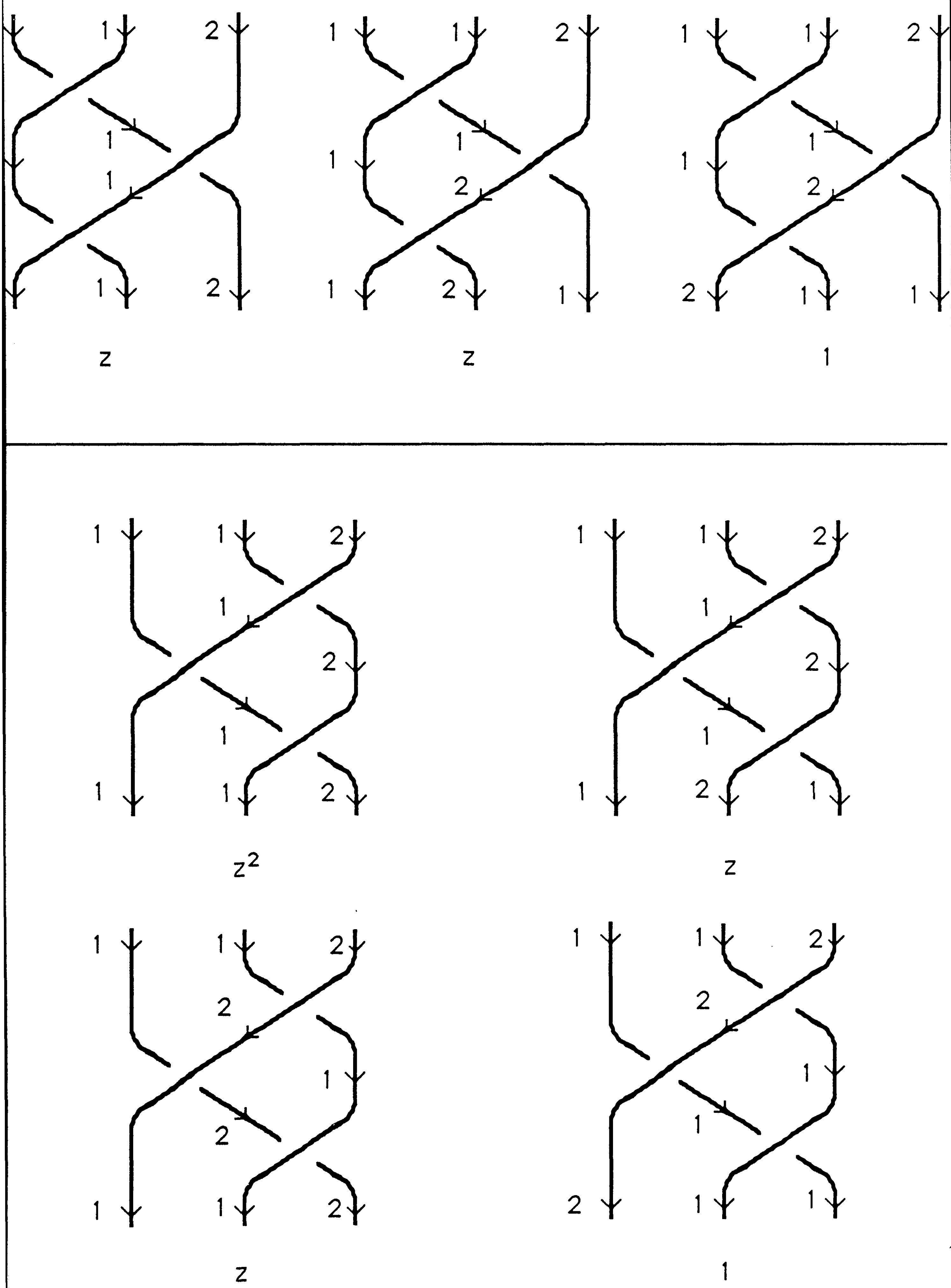


Figure 15

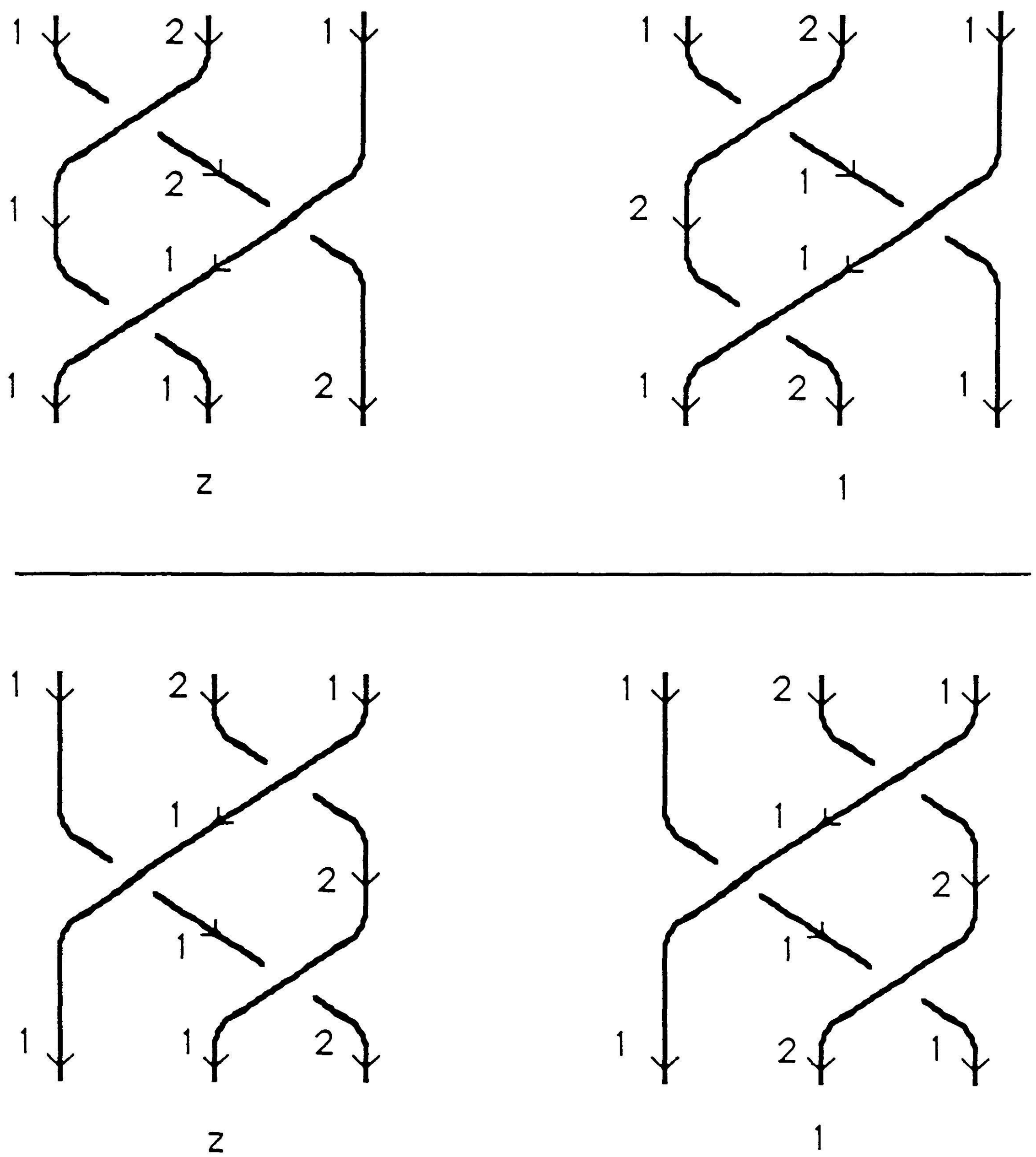


FIGURE 16

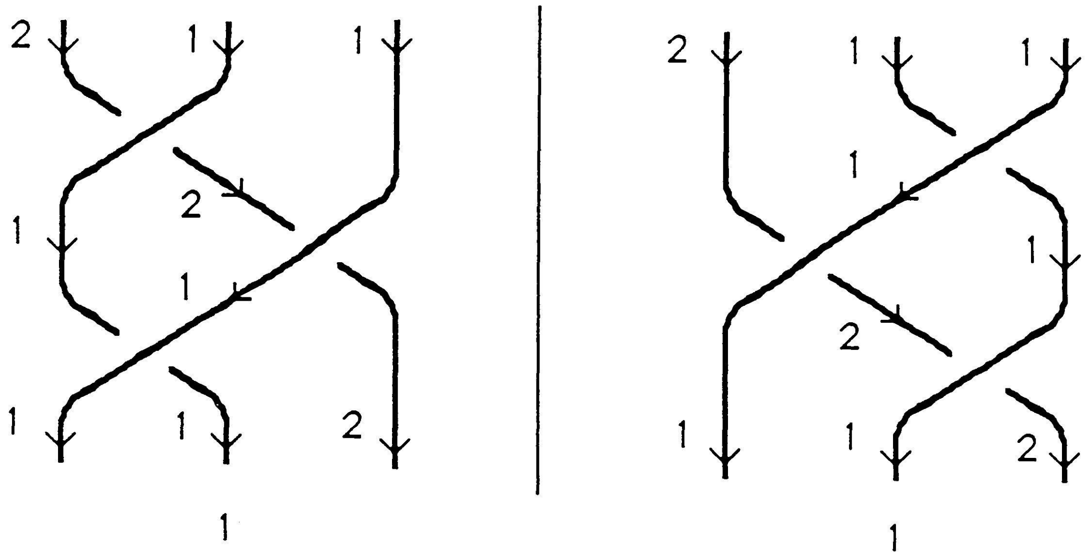


FIGURE 17

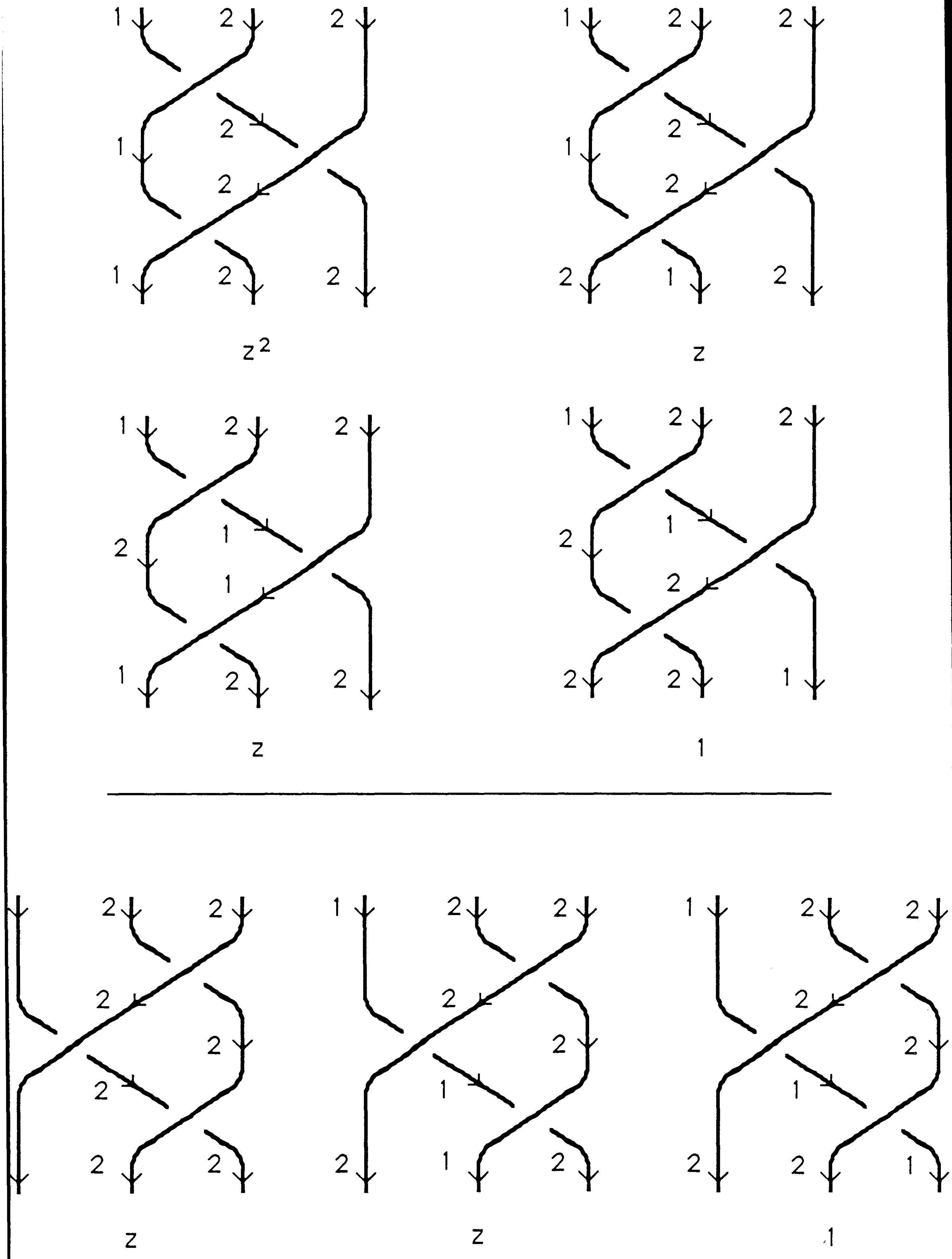


FIGURE 18

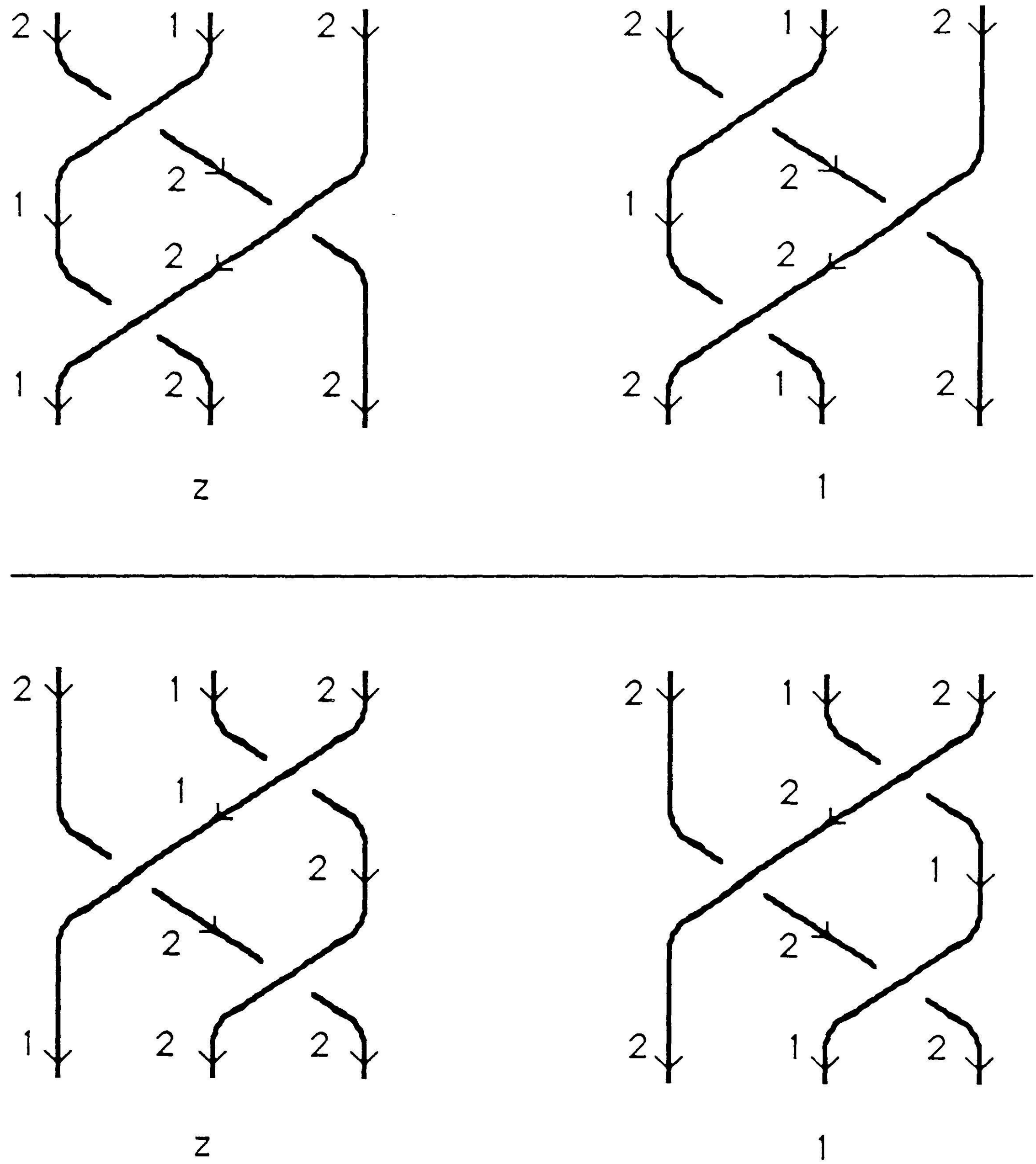


FIGURE 19

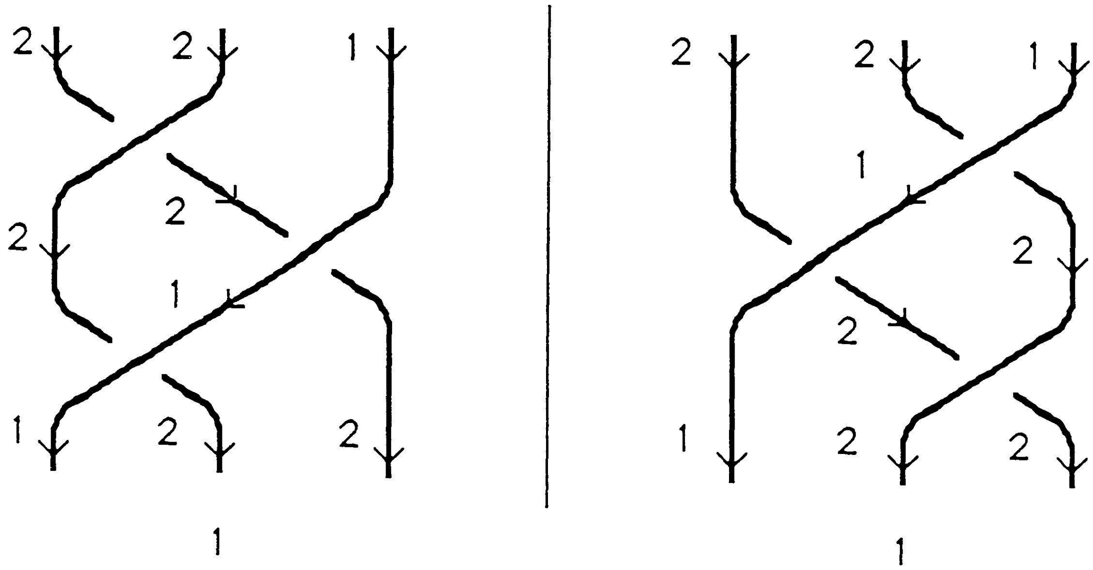


FIGURE 20

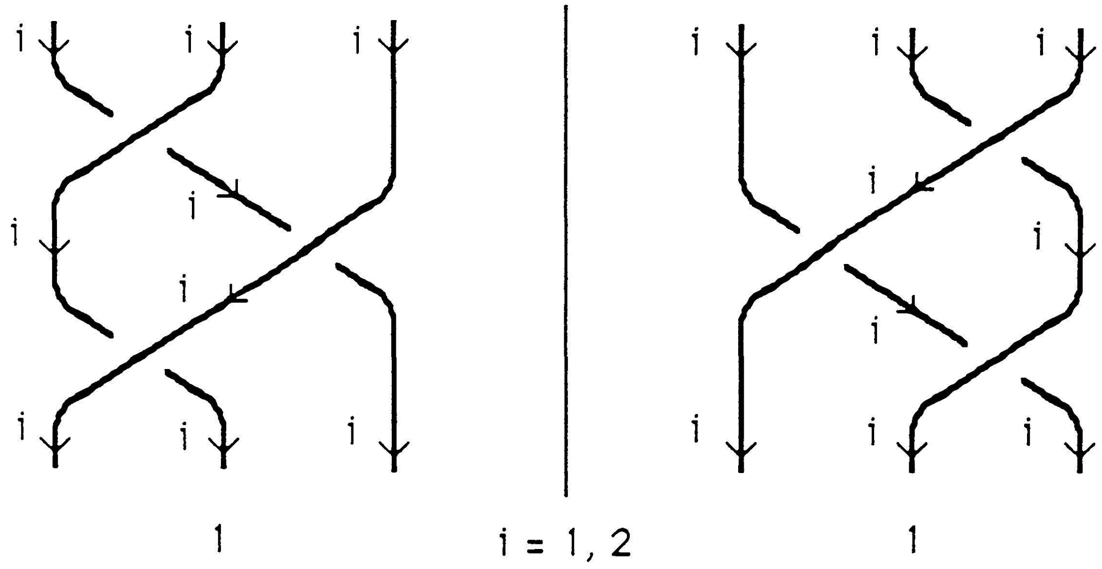


FIGURE 21

Clearly we may restrict our attention to the elements of $L(D, 2)$ and $L(D', 2)$ whose behavior inside the separator is depicted in one of the Figures 15 to 21. Thus it remains to perform the following analysis for each one of these figures: divide it into "subfigures" according to the labellings of the boundary edges which are oriented from the interior of the separator towards its exterior. Each subfigure will correspond to a certain class of outer labellings characterized by their value on the boundary edges. Then for each subfigure choose an appropriate reference and check that $C'(D, f) = C'(D', f)$. This is immediate in Figures 16, 17, 19, 20 (subdiagrams of D and D' are in bijective correspondence) and in Figure 21 (by property (i) for H'). In Figures 15 and 18 we must use the property of H' described in Figure 22, which is an immediate consequence of properties (i) and (iii).

This completes the proof.

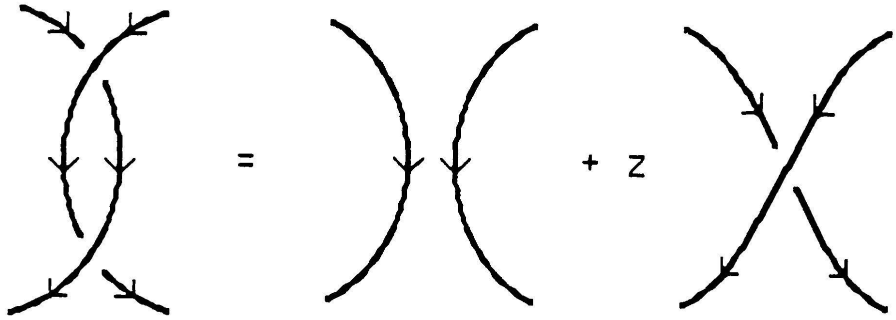


FIGURE 22

Remark. Figure 10 can be obtained from Figure 9 by a symmetry with respect to the vertical axis (without changing the crossing signs) together with the exchange of the numbers 1 and 2 in the labellings and the associated contributions. A similar relationship exists between Figures 13 and 14, 15 and 18, 16 and 19, 17 and 20 respectively. We could have used this to reduce the amount of case-checking in the proof of Proposition 1. However we found it simpler and more convincing to give the full set of figures.

2.3. A SEQUENCE OF STATE MODELS

We now derive from Proposition 1 state models for an infinite sequence of specializations of the homfly polynomial. This result appears in [20] and [28] where the original idea is attributed to [14].

We begin with a useful lemma.

Consider a labelling f of the diagram D together with a labelling g_i of $D_{f,i}$ for each i such that $f^{-1}(i)$ is not empty. We shall say that the labelling h of D is *compatible* with f and the g_i if for any two edges e, e' of D: if $f(e) < f(e')$ then $h(e) < h(e')$; if $f(e) = f(e') = i$ then $h(e) \leqslant h(e')$ if and only if $g_i(P_f(e)) \leqslant g_i(P_f(e'))$, where P_f is the projection associated to f (see Section 2.2).

UNIFICATION LEMMA. *For any labelling h of D compatible with f and the g_i,*

$$<D \mid h> \;=\; <D \mid f> \prod_i <D_{f,i} \mid g_i> .$$

This equality is easily proved by studying the possible contributions of a given vertex v to both sides. If v is incident only to edges labelled i by f, $<v \mid D \mid f> = 1$ and v is a vertex of $D_{f,j}$ if and only if $j = i$. Then the contribution of v to the right-hand side is $<v \mid D_{f,i} \mid g_i>$, which is clearly equal to $<v \mid D \mid h>$. If v is incident to edges labelled in two distinct ways by f, then v does not contribute to $\prod_i <D_{f,i} \mid g_i>$ and it is easy to check that $<v \mid D \mid h> = <v \mid D \mid f>$.

In the sequel we write $z = t - t^{-1}$.

PROPOSITION 2. *For any diagram D and positive integer q,*

$$H'(D, z, t^q) = t^{-(q+1)r(D)} \sum_{f \in L(D,q)} <D \mid f> t^{w(D,f)+2s(D,f)},$$

where $w(D, f) = \sum_{i=1,\dots q} w(D_{f,i})$ *and* $s(D, f) = \sum_{i=1,\dots q} ir(D_{f,i})$.

Proof. We proceed by induction on q.

For $q = 1$, $L(D, q)$ contains only one element f for which $<D \mid f> = 1$, $w(D, f) = w(D)$ and $s(D, f) = r(D)$. The result reduces to: $H'(D, z, t) = t^{w(D)}$. This is easy to check and well known.

Assume now that the result holds for the positive integer q. By Proposition 1

$$H'(D, z, t^{q+1}) = \sum_{f \in L(D, 2)} C(D, f),$$

with

$$C(D, f) = <D \mid f> t^{-r(D_{f,1})} t^{qr(D_{f,2})} H'(D_{f,1}, z, t^q) H'(D_{f,2}, z, t).$$

Let us fix f and write $D1$ for $D_{f,1}$, $D2$ for $D_{f,2}$.
By the induction hypothesis

$$H'(D1, z, t^q) = t^{-(q+1)r(D1)} \sum_{g \in L(D1, q)} <D1 \mid g> t^{w(D1, g) + 2s(D1, g)}$$

and we have seen that $H'(D2, z, t) = t^{w(D2)}$.
It follows that $C(D, f)$ is equal to

$$<D \mid f> t^{-r(D1) + qr(D2)} t^{-(q+1)r(D1)} \sum_{g \in L(D1, q)} <D1 \mid g> t^{w(D1, g) + 2s(D1, g)} t^{w(D2)}$$

Since $r(D1) + r(D2) = r(D)$, $C(D, f)$ can be rewritten as

$$t^{-(q+2)r(D)} \sum_{g \in L(D1, q)} <D \mid f> <D1 \mid g> t^{w(D1, g) + w(D2) + 2s(D1, g) + (2q+2)r(D2)}.$$

Now for every labelling g in $L(D1, q)$ define a labelling h of D as follows. For an edge e of D, if $f(e) = 1$ then $h(e) = g(P_f(e))$; if $f(e) = 2$ then $h(e) = q + 1$. The labelling h clearly belongs to $L(D, q+1)$ and we shall denote it by $u(f, g)$.

We first note that

$$w(D1, g) + w(D2) = w(D, u(f, g))$$

and

$$s(D1, g) + (q+1) r(D2) = s(D, u(f, g)).$$

Moreover $u(f, g)$ is compatible with f, the labelling g of $D1$ and the constant labelling of $D2$ with value $q + 1$. Hence, by the Unification Lemma, $<D \mid f> <D1 \mid g> = <D \mid u(f, g)>$.

Using the above remarks, $C(D, f)$ can be rewritten as

$$t^{-(q+2)r(D)} \sum_{g \in L(D1, q)} <D \mid u(f, g)> t^{w(D, u(f, g)) + 2s(D, u(f, g))}.$$

Since u is easily seen to define a bijection from

$$\{(f, g)/f \in L(D, 2), g \in L(D_{f,1}, q)\}$$

to $L(D, q+1)$ we obtain

$$H'(D, z, t^{q+1}) = t^{-(q+2)r(D)} \sum_{h \in L(D, q+1)} <D \mid h> t^{w(D, h) + 2s(D, h)}$$

as required.

Remarks. (1) The case $q = 2$ in Proposition 2 yields a state model for the Jones polynomial which, as noted in [28], can be directly related to Kauffman's "bracket polynomial" model [17, 18] via the theory of "ice-type models" developed in [2], Section 12.3.

(2) The state models of Proposition 2 can be used as shown in [28] to obtain a proof of the existence of the homfly polynomial.

(3) The applicability of Proposition 1 is limited by the fact that it cannot deal efficiently with the Alexander-Conway polynomial. This is because for every non-empty diagram D, $H'(D, z, 1) = 0$. Another aspect of this phenomenon is that property (v) cannot be defined in a coherent way for $H(D, z, 1) = A(D, z)$: the effect of adding a free loop to a non-empty diagram is qualitatively different from the corresponding effect on the empty diagram. However, a coherent version of property (v) is essential to the proof of Proposition 1. This has lead us to look for another form of the composition product which will be capable of handling the Alexander-Conway polynomial. So far we have been able to define such a composition product only in the case of closed braids. This is presented in the next section.

3. The specified composition product for closed braids

3.1. Braid words, braid diagrams and the specified product

Let us consider an infinite sequence of symbols (s_i) indexed by the set of positive integers. Artin's braid group on n strings $B_n (n \geqslant 1)$ can be defined by the presentation:

$$<s_1, \dots s_{n-1} \mid s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, i = 1, \dots n-2; s_i s_j = s_j s_i, |i-j| \geqslant 2> .$$

Thus B_1 is the trivial group and B_n is the subgroup of B_{n+1} generated by $s_1, \dots s_{n-1}$. We call *braid word on n strings* any word on the alphabet $\{s_i, s_i^{-1}/i = 1, \dots n-1\}$. Thus a braid word on n strings is also a braid word on n' strings for all $n' \geqslant n$. To every braid word m on n strings we